University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

Solutions to

FINAL EXAMINATION, JUNE, 2013

First Year - CHE, CIV, IND, LME, MEC, MSE

MAT187H1F - CALCULUS II

Exam Type: A

Examiner: D. Burbulla

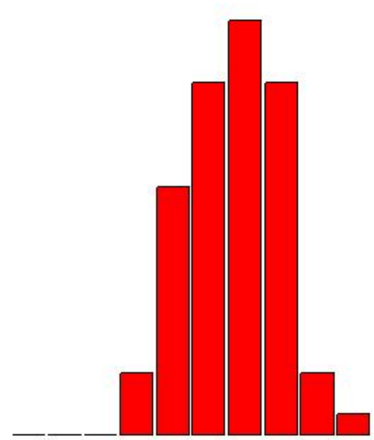
INSTRUCTIONS: Attempt all 9 questions; present your solutions in the exam books provided. You must show your work and give complete explanations to get full marks. The value of each question is indicated in parentheses beside the question number. Only aids permitted: Casio 260, Sharp 520 or TI 30 calculator. **Total Marks:** 100.

Comments:

1. The hard questions on this exam were Questions 3(a) and 7. The rest were routine.
2. Three of the most common inexplicable errors were: ignoring $x = 0$ in Question 1, simplifying $e^{3 \ln x/2}$ as $3x/2$ in Question 3(a), and not calculating the cross terms of $(2 + 2 \cos \theta)^2$ or $(2 + \sin(2\theta))^2$ in Question 7. Algebra is generally weak ....
3. Question 4(b) is an area, not an arc length, question! It can be done parametrically or by eliminating the parameter, but *nobody* got it completely correct.
4. The key to the polar graph of $r = 2 + \sin(2\theta)$ is to realize that $1 \leq r \leq 3$.

Breakdown of Results: All 73 registered students wrote this exam. The marks ranged from 34% to 90%, and the average was 61.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	1.4%
A	5.5%	80-89%	4.1%
B	23.3%	70-79%	23.3%
C	27.4%	60-69%	27.4%
D	23.3%	50-59%	23.3%
F	20.5%	40-49%	16.4%
		30-39%	4.1%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



1. [avg: 8.2/12] Find all the critical points of the function $f(x,y) = 2x^2 + y^3 - x^2y - 3y$, and determine if they are maximum points, minimum points, or saddle points.

Solution: Let $z = f(x,y)$. Then

$$\frac{\partial z}{\partial x} = 4x - 2xy \text{ and } \frac{\partial z}{\partial y} = 3y^2 - x^2 - 3.$$

Critical points:

$$\left.\begin{array}{rcl} 4x - 2xy & = & 0 \\ 3y^2 - x^2 & = & 3 \end{array}\right\} \Rightarrow \left\{ \begin{array}{c} x = 0 \text{ or } y = 2 \\ 3y^2 - 3 = x^2 \end{array}\right.$$

So the four critical points are

$$(x,y) = (0,1), (x,y) = (0,-1), (x,y) = (3,2) \text{ or } (x,y) = (-3,2).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 4 - 2y,\ \frac{\partial^2 z}{\partial y^2} = 6y,\ \frac{\partial^2 z}{\partial x \partial y} = -2x$$

and

$$\Delta = 24y - 12y^2 - 4x^2.$$

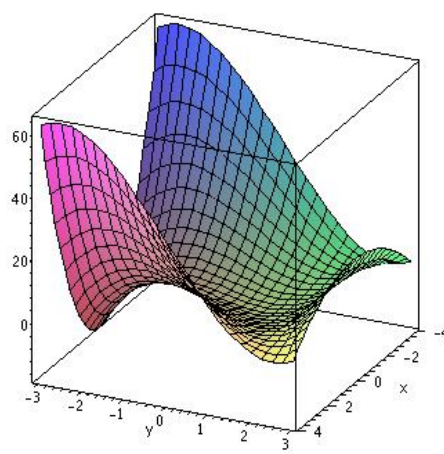
At each of $(0,-1)$ and $(\pm 3, 2)$ it turns out

$$\Delta = -36 < 0.$$

So f has a saddle point at each of these three locations. But at $(0,1)$

$$\Delta = 12 > 0, \frac{\partial^2 z}{\partial x^2} = 2 > 0,$$

so f has a min value at $(0,1)$.



2. [avg: 8.5/10] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

for some positive constant k, where T is the temperature of an object at time t in a room with constant ambient temperature A.

At 9:30 AM a freshly poured cup of coffee has temperature 180° F. After two minutes in a room with constant air temperature 70° F the coffee has cooled to 165° F. Find when the temperature of the coffee will be 120° F.

Solution: let $A = 70$ and separate variables. You can assume that $T > 70$.

$$\int \frac{dT}{T - 70} = \int -k\,dt \Rightarrow \ln(T - 70) = -kt + C.$$

Note: there is no need to solve this formula for T in general, but you could if you wanted.

Let t be measured in minutes and choose $t = 0$ to correspond to 9:30 AM. To find C take $t = 0, T = 180$:

$$\ln(180 - 70) = -k(0) + C \Rightarrow C = \ln 110.$$

To find k take $t = 2, T = 165$ and use $C = \ln 110$:

$$\ln(165 - 70) = -2k + \ln 110 \Rightarrow 2k = \ln 110 - \ln 95 = \ln(22/19) \Rightarrow k = \frac{1}{2}\ln(22/19).$$

To answer the question, let $T = 120$ and solve for t :

$$\begin{aligned} \ln(120 - 70) = -\frac{\ln(22/19)}{2}t + \ln 110 \quad &\Rightarrow \quad \frac{\ln(22/19)}{2}t = \ln 110 - \ln 50 = \ln 2.2 \\ &\Rightarrow \quad t = \frac{2\ln 2.2}{\ln(22/19)} \approx 10.75622572 \end{aligned}$$

So the temperature of the coffee will be 120° F at about 9:40:45 AM.

3. [avg: 7.2/12] Find the general solution for each of the following differential equations:

(a) [7 marks] $\dfrac{dy}{dx} + \dfrac{3y}{2x} = \dfrac{1}{1+x}$, for $x > 0$.

Solution: use the method of the integrating factor.

$$\mu = e^{\int \frac{3\,dx}{2x}} = e^{3\ln(x)/2} = x^{3/2}.$$

Then:

$$y = \frac{1}{\mu}\int \frac{\mu\,dx}{1+x} = x^{-3/2}\int \frac{x^{3/2}}{1+x}\,dx.$$

This integral requires some work: let $x = u^2, dx = 2u\,du$. Then

$$\begin{aligned}
\int \frac{x^{3/2}\,dx}{1+x} &= \int \frac{u^3\,2u}{1+u^2}du \\
&= 2\int \frac{u^4}{1+u^2}du \\
&= 2\int \left(u^2 - 1 + \frac{1}{1+u^2}\right) du \\
&= \frac{2u^3}{3} - 2u + 2\tan^{-1} u + C \\
&= \frac{2x^{3/2}}{3} - 2\sqrt{x} + 2\tan^{-1}\sqrt{x} + C
\end{aligned}$$

Finally,

$$y = x^{-3/2}\left(\frac{2x^{3/2}}{3} - 2\sqrt{x} + 2\tan^{-1}\sqrt{x} + C\right) = \frac{2}{3} - \frac{2}{x} + \frac{2\tan^{-1}\sqrt{x}}{x^{3/2}} + \frac{C}{x^{3/2}}.$$

(b) [5 marks] $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 13y = 0$

Solution:

$$r^2 + 6r + 13 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm 2i,$$

so

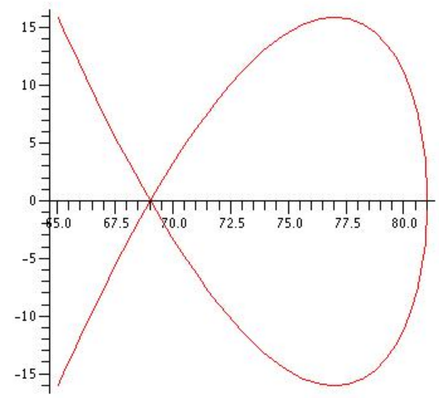
$$y = Ae^{-3x}\cos(2x) + Be^{-3x}\sin(2x).$$

4. [avg: 5.3/10] Suppose $x = 81 - t^2$ and $y = t^3 - 12t$ are the parametric equations of a curve in the xy-plane. Find the following:

(a) [5 marks] all points on the curve at which the tangent line to the curve is horizontal or vertical.

Solution:

$$\frac{dx}{dt} = -2t, \frac{dy}{dt} = 3t^2 - 12.$$



For horizontal tangents,

$$0 = \frac{dy}{dt} \Rightarrow t^2 = 4 \Rightarrow t = \pm 2,$$

and the points are $(x, y) = (77, \pm 16)$.
For vertical tangents,

$$0 = \frac{dx}{dt} \Rightarrow t = 0,$$

and the point is $(x, y) = (81, 0)$.

(b) [5 marks] the area enclosed by the curve for $69 \leq x \leq 81$.

Solution: $y \geq 0$ for $-\sqrt{12} \leq t \leq 0$. Then by symmetry,

$$\begin{aligned} A = 2\int_{69}^{81} y\,dx &= 2\int_{-\sqrt{12}}^{0} (t^3 - 12t)\,(-2t)\,dt = -4\int_{-\sqrt{12}}^{0} (t^4 - 12t^2)\,dt \\ &= -4\left[\frac{t^5}{5} - 4t^3\right]_{-\sqrt{12}}^{0} = 4\left(-\frac{144\sqrt{12}}{5} + 4(12\sqrt{12})\right) \\ &= 192\sqrt{12} - \frac{576}{5}\sqrt{12} = \frac{384}{5}\sqrt{12} \end{aligned}$$

Alternate Solution: Eliminate the parameter: for $y > 0$ pick $t = -\sqrt{81 - x}$. Then

$$\begin{aligned} A = 2\int_{69}^{81} y\,dx &= 2\int_{69}^{81} \left(-(81 - x)^{3/2} + 12\sqrt{81 - x}\right)\,dx \\ &= 2\left[\frac{2}{5}(81 - x)^{5/2} - 8(81 - x)^{3/2}\right]_{69}^{81} \\ &= \frac{384}{5}\sqrt{12} \end{aligned}$$

5. [avg: 6.8/12]

(a) [6 marks] Find the interval of convergence of the power series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{3^k(k+2)}$.

Solution: we have $a_k = \dfrac{(-1)^{k+1}}{3^k(k+2)}$, so

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{3^{k+1}(k+3)}{3^k(k+2)} = 3\lim_{k\to\infty}\frac{k+3}{k+2} = 3\times 1 = 3,$$

and the open interval of convergence is $(-R,R) = (-3,3)$. At the endpoint $x=3$ the series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k+2}$$

which converges by the alternating series test, since obviously the sequence $1/(k+2)$ decreases to zero as $k\to\infty$. But at $x=-3$ the series becomes

$$-\sum_{k=1}^{\infty}\frac{1}{k+2}$$

which diverges by the integral test, or by the limit comparison test with $b_k = 1/k$, the harmonic series. So the interval of convergence is $(-3,3]$.

(b) [6 marks] Use series to find an approximation of $\displaystyle\int_0^{1/2}\frac{dx}{\sqrt{1+x^4}}$ correct to within 10^{-5}.

Solution: use the binomial series, and integrate term by term:

$$\begin{aligned}
\int_0^{1/2}\frac{dx}{\sqrt{1+x^4}} &= \int_0^{1/2}(1+x^4)^{-1/2}\,dx \\
&= \int_0^{1/2}\left(1-\frac{x^4}{2}+\frac{3x^8}{8}-\frac{5x^{12}}{16}+\cdots\right)dx \\
&= \left[x-\frac{x^5}{10}+\frac{x^9}{24}-\frac{5x^{13}}{208}+\cdots\right]_0^{1/2} \\
&= \underbrace{\frac{1}{2}-\frac{1}{320}+\frac{1}{12288}}-\frac{5}{(208)(2^{13})}+\cdots \\
&\approx 0.4969563802,\ \text{correct to within } \frac{5}{(208)(2^{13})}\approx 2.9\times 10^{-6} < 10^{-5},
\end{aligned}$$

by the alternating series test remainder term.

6. [avg: 6.9/10] One player kicks a soccer ball to another player standing 25 m away with initial speed 18 m/sec. Find two angles at which the ball can be kicked. Which angle results in the ball getting to the other player the fastest? (Ignore air resistance; use $g = 9.8$ m/s^2.)

Solution: Let α be the angle of elevation from the ground at which the ball is kicked. Let the position of the ball at time t be (x, y) with the position at $t = 0$ set at $(0, 0)$. We have $v_0 = 18$ and

$$x = 18\cos(\alpha)\,t,\ \ y = 18\sin(\alpha)\,t - 4.9\,t^2.$$

The trajectory of the ball is to pass through the point $(x, y) = (25, 0)$ whence

$$25 = 18\cos(\alpha)\,t,\ \ 0 = 18\sin(\alpha)\,t - 4.9\,t^2.$$

Thus $t = 0$ or

$$t = \frac{18\sin\alpha}{4.9}.$$

Then

$$25 = \frac{18^2\cos\alpha\sin\alpha}{4.9} \Rightarrow \sin(2\alpha) = \frac{25(9.8)}{18^2} \Rightarrow 2\alpha \approx 49.1^\circ \text{ or } 130.9^\circ.$$

The two possible angles at which the ball can be kicked are

$$\alpha \approx 24.55^\circ \text{ or } 65.45^\circ.$$

The time taken in each case is

$$t = \frac{18\sin\alpha}{4.9} \approx 1.53 \text{ or } 3.34 \text{ sec},$$

so the angle $\alpha = 24.55^\circ$ results in the ball getting to the other player the fastest.

7. [avg: 5.5/12] For $0 \leq \theta \leq 2\pi$, sketch the curves r_1, r_2 with polar equations

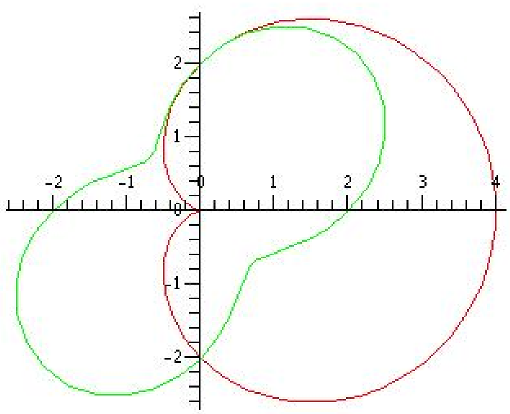
$$r_1 = 2 + \sin(2\theta);\ \ r_2 = 2 + 2\cos\theta,$$

and then find the area of the region outside the curve r_1 but inside the curve r_2.

Solution: at the intersection,

$$r_1 = r_2 \Leftrightarrow \sin(2\theta) = 2\cos\theta \Rightarrow 2\sin\theta\cos\theta = 2\cos\theta \Rightarrow \cos\theta = 0 \text{ or } \sin\theta = 1.$$

The solutions are $\theta = \pm\pi/2$.



In the diagram to the left, the green curve is the polar graph of

$$r_1 = 2 + \sin(2\theta),$$

and the red curve is the polar graph of

$$r_2 = 2 + 2\cos\theta.$$

The region in question is inside the red cardioid and outside the green peanut. Its area is given by

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (r_2^2 - r_1^2)\,d\theta$$

The calculations are:

$$\begin{aligned}
A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (r_2^2 - r_1^2)\,d\theta &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(4 + 8\cos\theta + 4\cos^2\theta - 4 - 4\sin(2\theta) - \sin^2(2\theta)\right)\,d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(8\cos\theta - 4\sin(2\theta) + 4\cos^2\theta - \sin^2(2\theta)\right)\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left(4\cos\theta - 2\sin(2\theta) + 1 + \cos(2\theta) - \frac{1}{4} + \frac{\cos(4\theta)}{4}\right)\,d\theta \\
&= \left[4\sin\theta + \cos(2\theta) + \frac{3\theta}{4} + \frac{\sin(2\theta)}{2} + \frac{\sin(4\theta)}{16}\right]_{-\pi/2}^{\pi/2} \\
&= 8 + \frac{3\pi}{4}
\end{aligned}$$

8. [avg: 3.3/10]

(a) [5 marks] Use the power series for e^x to show that

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots .$$

Solution: use

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and let $x = -1$. Then

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots \Leftrightarrow \frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots .$$

(b) [5 marks] Find all values of x for which the following power series converges:

$$f(x) = 1 + 3x + x^2 + 27x^3 + x^4 + 243x^5 + \cdots .$$

Solution: observe that $f(x)$ is the sum of two infinite geometric series:

$$f(x) = 1 + x^2 + x^4 + \cdots + 3x\,(1 + 9x^2 + 81x^4 + \cdots)$$

The infinite geometric series

$$1 + x^2 + x^4 + \cdots$$

converges if $x^2 < 1 \Leftrightarrow -1 < x < 1$. On the other hand, the infinite geometric series

$$1 + 9x^2 + (9x^2)^2 + (9x^2)^3 + \cdots$$

converges if

$$9x^2 < 1 \Leftrightarrow -\frac{1}{3} < x < \frac{1}{3}.$$

For $f(x)$ to converge both of these infinite geometric series must converge, that is

$$-\frac{1}{3} < x < \frac{1}{3}.$$

9. [avg: 9.7/12] Consider the curve with vector equation

$$\mathbf{r} = t^2\,\mathbf{i} + (\cos t + t\,\sin t)\,\mathbf{j} + (\sin t - t\,\cos t)\,\mathbf{k}$$

for $t \geq 0$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = 2t\,\mathbf{i} + t\,\cos t\,\mathbf{j} + t\,\sin t\,\mathbf{k};$$

$$\begin{aligned}\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{4t^2 + t^2\cos^2 t + t^2\sin^2 t} \\ &= \sqrt{4t^2 + t^2} \\ &= \sqrt{5t^2} \\ &= t\sqrt{5},\ \text{since } t \geq 0\end{aligned}$$

(b) [6 marks] Find an arc length parameterization of the curve, with reference point $(0, 1, 0)$, for which $t = 0$.

Solution: for $t \geq 0$,

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \int_0^t u\sqrt{5}\,du = \left[\frac{u^2\sqrt{5}}{2}\right]_0^t = \frac{t^2\sqrt{5}}{2} \Rightarrow t = \sqrt{\frac{2\,s}{\sqrt{5}}}.$$

So an arc length parametrization of the curve is

$$\mathbf{r} = \frac{2\,s}{\sqrt{5}}\,\mathbf{i} + \left(\cos\sqrt{\frac{2\,s}{\sqrt{5}}} + \sqrt{\frac{2\,s}{\sqrt{5}}}\sin\sqrt{\frac{2\,s}{\sqrt{5}}}\right)\mathbf{j} + \left(\sin\sqrt{\frac{2\,s}{\sqrt{5}}} - \sqrt{\frac{2\,s}{\sqrt{5}}}\cos\sqrt{\frac{2\,s}{\sqrt{5}}}\right)\mathbf{k}.$$