University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, JUNE, 2012

First Year - CHE, CIV, IND, LME, MEC, MSE

MAT187H1F - CALCULUS II

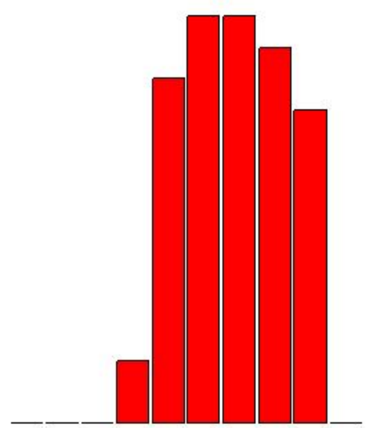
Exam Type: A

Examiner: D. Burbulla

INSTRUCTIONS: Attempt all 8 questions; present your solutions in the exam books provided. You must show your work and give complete explanations to get full marks. The value of each question is indicated in parentheses beside the question number. Only aids permitted: Casio 260, Sharp 520 or TI 30 calculator. **Total Marks:** 100.

Breakdown of Results: 61 registered students wrote this exam. The marks ranged from 34% to 89%, and the average was 63.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	16.4%	80-89%	16.4%
B	19.7%	70-79%	19.7%
C	21.3%	60-69%	21.3%
D	21.3%	50-59%	21.3%
F	21.3%	40-49%	18.0%
		30-39%	3.3%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



1. [15 marks; 5 for each part. Avg: 12.7] Find the general solution, y as a function of x, to each of the following differential equations:

   (a) $\dfrac{dy}{dx} = \dfrac{\cos x}{e^y}$

   Solution: separate variables.

$$\int e^y\,dy = \int \cos x\,dx \Rightarrow e^y = \sin x + C.$$

   So

$$y = \ln\left(\sin x + C\right).$$

   (b) $\dfrac{dy}{dx} - \dfrac{y}{x} = x$

   Solution: use the method of the integrating factor.

$$\mu = e^{-\int \frac{dx}{x}} = e^{-\ln|x|} = \frac{1}{|x|} \Rightarrow \mu = \frac{1}{x} \text{ or } \mu = -\frac{1}{x}.$$

   Taking the former:

$$y = \frac{1}{\mu}\int \mu\,x\,dx = x\int 1\,dx = x(x+C) = x^2 + Cx.$$

   (c) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 0$

   Solution:

$$r^2 + 6r + 10 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i,$$

   so

$$y = Ae^{-3x}\cos x + Be^{-3x}\sin x.$$

2. [12 marks; 6 for each part. Avg: 6.2] Find the following:

(a) the interval of convergence of the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k(k+1)}$.

Solution: we have $a_k = \dfrac{1}{k(k+1)}$, so

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{(k+1)(k+2)}{k(k+1)} = \lim_{k\to\infty} \frac{(k+2}{k} = 1,$$

and the open interval of convergence is $(2-R, 2+R) = (1,3)$. At the endpoints, $x = 1, 3$ both series

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(k+1)}$$

converge absolutely by comparison with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

for which $p = 2 > 1$. Thus the interval of convergence is $[1, 3]$.

(b) the first three non-zero terms in the Maclaurin series for $\ln\left(\dfrac{1+x}{1-x}\right)$.

Solution: $\ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$, so

$$\begin{aligned}
&\ln\left(\frac{1+x}{1-x}\right) \\
&= \ln(1+x) - \ln(1-x) \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots - \left(-x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \cdots\right) \\
&= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} - \cdots
\end{aligned}$$

Alternate Solution: let $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$. Then

$$f'(x) = \frac{(1-x)}{(1+x)} \frac{(1-x+1+x)}{(1-x)^2} = \frac{2}{1-x^2} = 2 + 2x^2 + 2x^4 + \cdots;$$

so

$$f(x) = \int (2 + 2x^2 + 2x^4 + \cdots)\, dx = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

3. [15 marks. Avg: 10.3] Find the following:

(a) [8 marls] $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(x,y)=(\pi,\pi)$ if $x=\pi+\sin t, y=t+\sin^2 t$.

Solution: $(x,y)=(\pi,\pi) \Rightarrow t=\pi$.

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1+2\sin t\,\cos t}{\cos t}=\sec t+2\sin t;$$

$$\frac{d^2y}{dx^2}=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}=\frac{\sec t\tan t+2\cos t}{\cos t}.$$

So at the point $(x,y)=(\pi,\pi)$, for which $t=\pi$, we have

$$\frac{dy}{dx}=-1 \text{ and } \frac{d^2y}{dx^2}=2.$$

(b) [7 marks] the approximate value of $\displaystyle\int_0^{1/2}\tan^{-1}(x^2)\,dx$ correct to within 10^{-5}, and explain how you know your answer is correct to within 10^{-5}.

Solution:

$$\begin{aligned}\int_0^{1/2}\tan^{-1}(x^2)\,dx &= \int_0^{1/2}\left(x^2-\frac{(x^2)^3}{3}+\frac{(x^2)^5}{5}-\cdots\right)dx\\ &= \int_0^{1/2}\left(x^2-\frac{x^6}{3}+\frac{x^{10}}{5}-\cdots\right)dx\\ &= \left[\frac{x^3}{3}-\frac{x^7}{21}+\frac{x^{11}}{55}-\cdots\right]_0^{1/2}\\ &= \frac{1}{24}-\frac{1}{2688}+\frac{1}{112640}-\cdots\\ &\approx 0.041294642\ldots\end{aligned}$$

correct to within

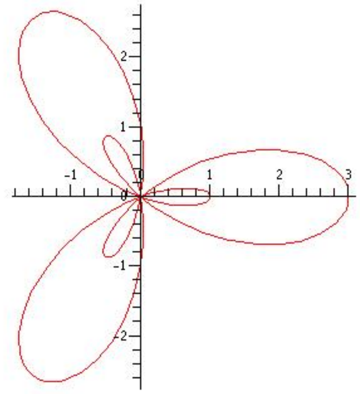
$$\frac{1}{112640}=0.000008877\cdots<10^{-5},$$

by the alternating series remainder formula.

4. [12 marks. Avg: 6.2] Let

$$r = 1 + 2\cos(3\theta),$$

which has polar graph shown to the right. Find the area of the region which is inside the three large loops but outside the three small loops.



Solution: all the loops emanate from the origin.

$$r = 0 \Leftrightarrow 1 + 2\cos(3\theta) = 0 \Leftrightarrow \cos(3\theta) = -\frac{1}{2} \Leftrightarrow 3\theta = \pm\frac{2\pi}{3} + 2n\pi \Leftrightarrow \theta = \pm\frac{2\pi}{9} + \frac{2n\pi}{3}.$$

So the limits of the big loop with $x > 0$ are $\theta = \pm 2\pi/9$, and the limits for the small loop in the third quadrant are $\theta = 2\pi/9$ and $\theta = 4\pi/9$. Then the area within one big loop is

$$A_1 = \frac{1}{2}\int_{-2\pi/9}^{2\pi/9} (1 + 2\cos(3\theta))^2 \, d\theta \text{ or } \int_0^{2\pi/9} (1 + 2\cos(3\theta))^2 \, d\theta,$$

and the area within one small loop is

$$A_2 = \frac{1}{2}\int_{2\pi/9}^{4\pi/9} (1 + 2\cos(3\theta))^2 \, d\theta \text{ or } \int_{\pi/3}^{4\pi/9} (1 + 2\cos(3\theta))^2 \, d\theta.$$

In each case, use $\int (1 + 2\cos(3\theta))^2 \, d\theta =$

$$\int (1+4\cos(3\theta)+4\cos^2(3\theta)) \, d\theta = \int (3+4\cos(3\theta)+2\cos(6\theta)) \, d\theta = 3\theta + \frac{4}{3}\sin(3\theta) + \frac{1}{3}\sin(6\theta) + C.$$

So the area within the big loops but outside the small loops is

$$A = 3A_1 - 3A_2 = 3\left[3\theta + \frac{4}{3}\sin(3\theta) + \frac{1}{3}\sin(6\theta)\right]_0^{2\pi/9} - 3\left[3\theta + \frac{4}{3}\sin(3\theta) + \frac{1}{3}\sin(6\theta)\right]_{\pi/3}^{4\pi/9}$$

$$= 3\left(\frac{2\pi}{3} + \frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{6}\right) - 3\left(\frac{4\pi}{3} - \pi - \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{6}\right) = 3\sqrt{3} + \pi.$$

5. [12 marks. Avg: 9.4] Find all the critical points of the function $f(x, y) = x^2y - 6y^2 - 3x^2$ and determine if they are maximum points, minimum points, or saddle points.

Solution: Let $z = f(x, y)$. Then

$$\frac{\partial z}{\partial x} = 2xy - 6x \text{ and } \frac{\partial z}{\partial y} = x^2 - 12y.$$

Critical points:

$$\left.\begin{array}{rcl} 2xy - 6x & = & 0 \\ x^2 - 12y & = & 0 \end{array}\right\} \Rightarrow \left\{ \begin{array}{c} x = 0 \text{ or } y = 3 \\ x^2 = 12y \end{array}\right.$$

So the three critical points are

$$(x, y) = (0, 0), (x, y) = (6, 3) \text{ or } (x, y) = (-6, 3).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 2y - 6, \ \frac{\partial^2 z}{\partial y^2} = -12, \ \frac{\partial^2 z}{\partial x \partial y} = 2x$$

and

$$\Delta = 72 - 24y - 4x^2.$$

At the critical points $(x, y) = (\pm 6, 3)$,

$$\Delta = -144 < 0$$

so f has a saddle point at both the critical points $(x, y) = (\pm 6, 3)$.

At the critical point $(x, y) = (0, 0)$,

$$\Delta = 72 > 0 \text{ and } \frac{\partial^2 z}{\partial y^2} = -12 < 0,$$

so there f has a maximum point at $(0, 0, 0)$.

6. [10 marks. Avg: 6.8] Find two elevation angles that will enable a shell, fired from ground level with a muzzle speed of 250 m/sec to hit a ground-level target 3 km away. Note: to simplify calculations let $g = 10$ m/sec^2.

Solution: Let α be the angle the initial velocity vector $\mathbf{v}_0$ makes with the x-axis. We have $v_0 = \|\mathbf{v}_0\| = 250$ and $\mathbf{a} = -10\,\mathbf{j}$; $\mathbf{v_0} = 250\cos\alpha\,\mathbf{i} + 250\sin\alpha\,\mathbf{j}$.

So

$$\mathbf{v} = \int \mathbf{a}\,dt = -10t\,\mathbf{j} + \mathbf{v_0} = 250\cos\alpha\,\mathbf{i} + (250\sin\alpha - 10t)\,\mathbf{j}$$

and, taking $\mathbf{r}_0 = 0\,\mathbf{i} + 0\,\mathbf{j}$,

$$\mathbf{r} = \int \mathbf{v}\,dt = 250\,t\,\cos\alpha\,\mathbf{i} + (250\,t\,\sin\alpha - 5t^2)\,\mathbf{j}.$$

That is,

$$x = 250\,t\,\cos\alpha \text{ and } y = 250\,t\,\sin\alpha - 5t^2.$$

We want the trajectory to pass through the point $(x, y) = (3000, 0)$, so

$$3000 = 250\,t\,\cos\alpha,\ \ 0 = 250\,t\,\sin\alpha - 5t^2.$$

Thus $t = 0$ or $t = 50\sin\alpha$, and so

$$3000 = 250(50\sin\alpha)\cos\alpha \Leftrightarrow \sin(2\alpha) = 0.48.$$

Therefore

$$2\alpha \approx 28.68^\circ \text{ or } 151.32^\circ$$

whence

$$\alpha \approx 14.34^\circ \text{ or } 75.66^\circ.$$

7. [12 marks. Avg: 4.4] Let $f(x) = \dfrac{x^2}{(1+x^2)^2}$.

(a) [6 marks] Use the binomial series expansion for $(1+x^2)^{-2}$ to find the Maclaurin series for $f(x)$.

Solution: use the binomial series

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots$$

with $z = x^2$ and $\alpha = -2$. Then

$$\begin{aligned} f(x) &= x^2(1+x^2)^{-2} \\ &= x^2\left(1 - 2x^2 + \frac{(-2)(-3)}{2!}(x^2)^2 + \frac{(-2)(-3)(-4)}{3!}(x^2)^3 + \cdot\right) \\ &= x^2\left(1 - 2x^2 + 3x^4 - 4x^6 + \cdots\right) \\ &= x^2 - 2x^4 + 3x^6 - 4x^8 + \cdots \\ &= x^2 - 2(x^2)^2 + 3(x^2)^3 - 4(x^2)^4 + \cdots \\ &= \sum_{n=1}^{\infty}(-1)^{n+1} n\,(x^2)^n \end{aligned}$$

(b) [6 marks] Find the exact value of $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{9^n}$. Hint: use part (a).

Solution: we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{9^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(3^2)^n} = f(1/3) = \frac{(1/3)^2}{(1+(1/3)^2)^2} = 0.09$$

8. [12 marks. Avg: 7.3] The vector equation of a cycloid is

$$\mathbf{r} = (a\,t - a\,\sin t)\,\mathbf{i} + (a - a\,\cos t)\,\mathbf{j}.$$

for $a > 0$ and $0 \le t \le 2\pi$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = (a - a\,cost)\,\mathbf{i} + a\,\sin t\,\mathbf{j};$$

$$\begin{aligned}\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{(a - a\,cost)^2 + a^2\,\sin^2 t}\\ &= \sqrt{a^2(1 - 2\cos t + \cos^2 t) + a^2 \sin^2 t}\\ &= a\sqrt{2 - 2\cos t}\\ &= a\,\sqrt{4\sin^2(t/2)}\\ &= 2a\sin(t/2),\end{aligned}$$

since $0 \le t \le 2\pi \Rightarrow 0 \le t/2 \le \pi \Rightarrow \sin(t/2) \ge 0$.

(b) [6 marks] Find an arc length parametrization of the cycloid, with reference point $(0,0)$, for which $t = 0$.

Solution:

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = [-4a\cos(u/2)]_0^t = 4a - 4a\cos(t/2).$$

Then $0 \le s \le 8a$ and

$$\cos(t/2) = \frac{4a - s}{4a} \Rightarrow t = 2\cos^{-1}\left(\frac{4a - s}{4a}\right),$$

so an arc length parametrization of the curve is $\mathbf{r} =$

$$\left(2a\cos^{-1}\left(\frac{4a-s}{4a}\right) - a\sin\left(2\cos^{-1}\left(\frac{4a-s}{4a}\right)\right)\right)\mathbf{i} + \left(a - a\cos\left(2\cos^{-1}\left(\frac{4a-s}{4a}\right)\right)\right)\mathbf{j}.$$