University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, JUNE, 2010**
First Year - CHE, CIV, IND, LME, MEC, MSE

MAT187H1F - CALCULUS II
Exam Type: A

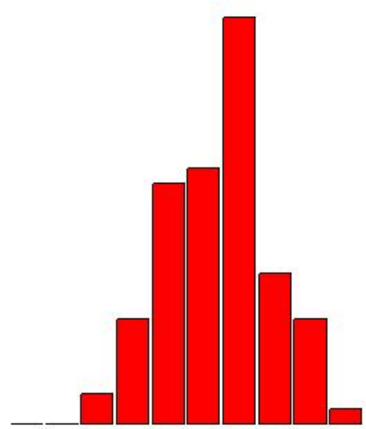
Comments:

1. Many students displayed their poor—very poor—algebraic skills on this exam; and many of these students failed the exam because of their poor algebra. There is no excuse for students who think that the square root function is linear, or who think that the reciprocal of a sum is the sum of the reciprocals.
2. Lots of students had no idea how to handle the absolute value signs in applying the root test to Question 4, or in solving for the interval of convergence in Question 2.
3. It's a major faux pas to solve Question 3 by using the definition of a Maclaurin series. Instead, combine series that you (should) already know!
4. Question 5 was probably the hardest question on the exam, in the sense that the integral of $\sqrt{1+\sin\theta}$ is quite tricky. However, one student came up with a very nice solution, different from the one I used in the solutions:

$$\sqrt{1+\sin\theta} = \sqrt{\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}} = \sqrt{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2}.$$

Breakdown of Results: All 87 registered students wrote this exam. The marks ranged from 27% to 91%, and the average was 58.1%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	1.2%
A	9.3%	80-89%	8.1%
B	11.5%	70-79%	11.5%
C	31.0%	60-69%	31.0%
D	19.5 %	50-59%	19.5%
F	28.7%	40-49%	18.4%
		30-39%	8.0%
		20-29%	2.3%
		10-19%	0.0%
		0-9%	0.0%



1. [15 marks] Solve the following initial value problems for y as a function of x:

(a) [5 marks] DE: $(1+x^4)\,\dfrac{dy}{dx} = \dfrac{x^3}{e^y}$;   IC: $y(0) = 1$.

Solution: separate variables.

$$\int e^y\,dy = \int \frac{x^3\,dx}{1+x^4} \Rightarrow e^y = \frac{1}{4}\ln(1+x^4) + C.$$

Using the initial condition: $y(0) = 1 \Rightarrow C = e$. So

$$y = \ln\left(\frac{1}{4}\ln(1+x^4) + e\right).$$

(b) [5 marks] DE: $\dfrac{dy}{dx} - y = e^{2x}$;   IC: $y(0) = 3$.

Solution:

$$\begin{aligned} \mu = e^{-\int dx} = e^{-x} \Rightarrow y &= \frac{1}{\mu}\int \mu\, e^{2x}\,dx \\ &= e^x\left(\int e^x\,dx\right) \\ &= e^x\,(e^x + C) \\ &= e^{2x} + C\,e^x \\ y(0) = 3 \Rightarrow C = 2,\ \text{so } y &= e^{2x} + 2e^x. \end{aligned}$$

(c) [5 marks] DE: $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$;   IC: $y(0) = 2$; $y'(0) = 7$.

Solution: $r^2 + 5r + 6 = 0 \Rightarrow r = -2$ or -3, so $y = Ae^{-2x} + Be^{-3x}$, and hence $y' = -2A\,e^{-2x} - 3B\,e^{-3x}$. Then

$$\left.\begin{array}{rcl} y(0) = 2 & \Rightarrow & 2 = A + B \\ y'(0) = 7 & \Rightarrow & 7 = -2A - 3B \end{array}\right\} \Rightarrow \quad A = 13,\ B = -11$$

So

$$y = 13\,e^{-2x} - 11\,e^{-3x}.$$

2. [5 marks] What is the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k+1}$?

Solution: $a_k = \frac{1}{k+1}$; $R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}} = \lim_{k\to\infty} \frac{k+2}{k+1} = 1$; so the open interval of convergence is $(-R+2, 2+R) = (1, 3)$. At $x = 1$, the series is

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+1},$$

which converges by the Alternating Series Test. At $x = 3$, the series is

$$\sum_{k=1}^{\infty} \frac{1}{k+1},$$

which diverges by the integral test. So the interval of convergence is $[1, 3)$.

3. [5 marks] Find the first three non-zero terms in the Maclaurin series for $e^{-x^2} \tan^{-1} x$.

Solution:

$$\begin{aligned} e^{-x^2} \tan^{-1} x &= \left(1 - x^2 + \frac{x^4}{2} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) \\ &= x - x^3 - \frac{x^3}{3} + \frac{x^5}{2} + \frac{x^5}{3} + \frac{x^5}{5} + \cdots \\ &= x - \frac{4}{3}x^3 + \frac{31}{30}x^5 + \cdots \end{aligned}$$

4. [5 marks] Determine whether the infinite series

$$\sum_{k=2}^{\infty} \left(-\frac{1}{\ln k}\right)^k$$

converges conditionally, converges absolutely, or diverges.

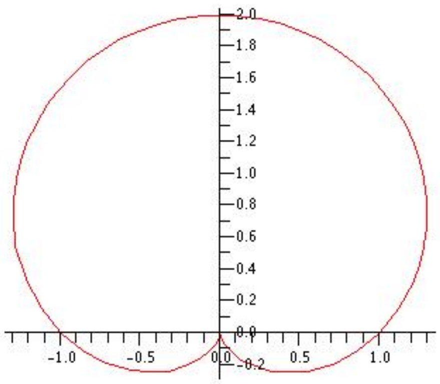
Solution: use the root test for absolute convergence.

$$\lim_{k\to\infty} \sqrt[k]{\left|\left(-\frac{1}{\ln k}\right)^k\right|} = \lim_{k\to\infty} \left(\frac{1}{\ln k}\right) = 0 < 1,$$

so the series converges absolutely.

5. [6 marks] Find the total arc length of the cardioid with polar equation $r = 1 + \sin\theta$.

Solution: by symmetry you can double the length of the right half.



$$\begin{aligned} L &= 2\int_{-\pi/2}^{\pi/2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \\ &= 2\int_{-\pi/2}^{\pi/2} \sqrt{2 + 2\sin\theta}\, d\theta \end{aligned}$$

This integral is quite tricky!

$$\begin{aligned} L &= 2\int_{-\pi/2}^{\pi/2} \sqrt{2 + 2\sin\theta}\, d\theta \\ &= 2\int_{-\pi/2}^{\pi/2} \sqrt{\frac{(2+2\sin\theta)(2-2\sin\theta)}{2-2\sin\theta}}\, d\theta \\ &= 2\int_{-\pi/2}^{\pi/2} \sqrt{\frac{4\cos^2\theta}{2-2\sin\theta}}\, d\theta \\ &= \frac{4}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta, \text{ since } \cos\theta > 0 \end{aligned}$$

Now let $u = 1 - \sin\theta$:

$$\begin{aligned} L &= 2\sqrt{2}\int_{2}^{0} \left(-\frac{1}{\sqrt{u}}\right) du \\ &= 2\sqrt{2}\int_{0}^{2} \frac{du}{\sqrt{u}} \\ &= 2\sqrt{2}\left[2\sqrt{u}\right]_0^2 \\ &= 8 \end{aligned}$$

6. [12 marks] At time $t = 0$ a baseball that is 1 m above the ground is hit with a baseball bat. The ball leaves the bat with a speed of 20 m/sec at an angle of 30° above the horizontal. Assume that the acceleration due to gravity is 9.8 m/sec^2 and neglect air resistance.

(a) [10 marks] How long will it take for the baseball to hit the ground? Express your answer to the nearest hundredth of a second.

Solution: $\mathbf{a} = -9.8\,\mathbf{j}$; $\mathbf{v_0} = 20\cos 30^\circ\,\mathbf{i} + 20\sin 30^\circ\,\mathbf{j} = 10\sqrt{3}\,\mathbf{i} + 10\,\mathbf{j}$; $\mathbf{r_0} = \mathbf{j}$.
So

$$\mathbf{v} = \int \mathbf{a}\,dt = -9.8t\,\mathbf{j} + \mathbf{v_0} = 10\sqrt{3}\,\mathbf{i} + (10 - 9.8t)\,\mathbf{j}$$

and

$$\mathbf{r} = \int \mathbf{v}\,dt = (10\sqrt{3})t\,\mathbf{i} + \left(10t - 4.9t^2\right)\,\mathbf{j} + \mathbf{r_0} = (10\sqrt{3})t\,\mathbf{i} + \left(1 + 10t - 4.9t^2\right)\,\mathbf{j}.$$

That is,

$$x = 10t\sqrt{3} \text{ and } y = 1 + 10\,t - 4.9\,t^2.$$

When the ground hits the ball $y = 0$ and $t > 0$; so

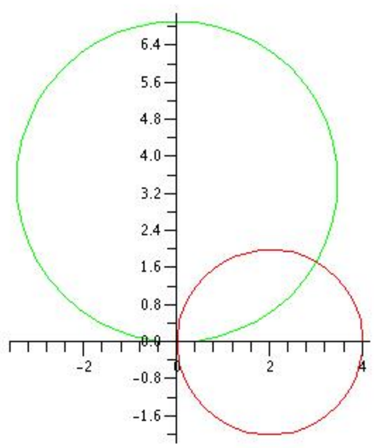
$$\begin{aligned} y = 0 \quad &\Rightarrow \quad 4.9\,t^2 - 10\,t - 1 = 0 \\ &\Rightarrow \quad t = \frac{10 \pm \sqrt{100 + 19.6}}{9.8} \\ &\Rightarrow \quad t \simeq 2.14, \text{ to the nearest hundredth of a second} \end{aligned}$$

(b) [2 marks] Find the horizontal distance travelled by the ball. Express your answer to the nearest tenth of a meter.

Solution: at $t = 2.14$, $x = (10\sqrt{3})\,(2.14) \simeq 37.1$

7. [12 marks] Find the area of the region that is inside both the circle with polar equation $r = 4\cos\theta$ and the circle with polar equation $r = 4\sqrt{3}\sin\theta$.

Comments: at the intersection, $4\cos\theta = 4\sqrt{3}\sin\theta \Rightarrow \tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \dfrac{\pi}{6}$.



The region in question is inside both the red circle and the green circle. Since the two circles have different radii, the region is *not* symmetric about the ray $\theta = \pi/6$. Thus the area of this region can't be calculated with one integral.
You need to know that the green circle, $r = 4\sqrt{3}\sin\theta$, passes through the origin at $\theta = 0$; and that the red circle, $r = 4\cos\theta$, passes through the origin at $\theta = \pi/2$.

Solution: you have to use double angle formulas.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/6} \left(4\sqrt{3}\sin\theta\right)^2 \, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (4\cos\theta)^2 \, d\theta \\
&= \frac{1}{2}\int_0^{\pi/6} 48\sin^2\theta \, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} 16\cos^2\theta \, d\theta \\
&= 12\int_0^{\pi/6} (1-\cos(2\theta)) \, d\theta + 4\int_{\pi/6}^{\pi/2} (1+\cos(2\theta)) \, d\theta \\
&= [12\theta - 6\sin(2\theta)]_0^{\pi/6} + [4\theta + 2\sin(2\theta)]_{\pi/6}^{\pi/2} \\
&= 2\pi - 3\sqrt{3} + 2\pi - \frac{2\pi}{3} - \sqrt{3} \\
&= \frac{10}{3}\pi - 4\sqrt{3}
\end{aligned}$$

8. [10 marks] Approximate

$$\int_0^{0.2} x^2\,(1+x^4)^{1/3}\,dx$$

to within 10^{-5}, and explain why your approximation is correct to within 10^{-5}.

Solution: use the binomial series

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots$$

with $z = x^4$ and $\alpha = 1/3$. Then

$$\begin{aligned}
&\int_0^{0.2} x^2\,(1+x^4)^{1/3}\,dx \\
=\ &\int_0^{0.2} x^2 \left(1 + \frac{1}{3}x^4 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(x^4)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{4}{3}\right)}{3!}(x^4)^3 + \cdots\right) dx \\
=\ &\int_0^{0.2} \left(x^2 + \frac{1}{3}x^6 - \frac{1}{9}x^{10} + \frac{4}{81}x^{14} - \cdots\right) dx \\
=\ &\left[\frac{x^3}{3} + \frac{1}{21}x^7 - \frac{1}{99}x^{11} + \frac{4}{1215}x^{15} - \cdots\right]_0^{0.2} \\
=\ &0.0026666666\cdots + 0.000000609\ldots\ - 2.07\times 10^{-10} + \cdots \\
=\ &0.002667275\ldots
\end{aligned}$$

which is correct to within $2.07\times 10^{-10} < 10^{-5}$, by the alternating series remainder term. You *must* refer to the alternating series remainder term to get full marks. Also, note that the first two terms are positive, so the series doesn't start alternating until the third term.

9. [10 marks] Find and classify all the critical points of the function $f(x, y) = x^2 + y^2 - \dfrac{2}{xy}$.

Solution: Let $z = f(x, y)$, for $xy \neq 0$. Then

$$\frac{\partial z}{\partial x} = 2x + \frac{2}{x^2 y} \text{ and } \frac{\partial z}{\partial y} = 2y + \frac{2}{xy^2}.$$

Critical points:

$$\left.\begin{array}{rcl} 2x + \dfrac{2}{x^2 y} & = & 0 \\ 2y + \dfrac{2}{xy^2} & = & 0 \end{array}\right\} \Rightarrow \left.\begin{array}{rcl} x^3 y & = & -1 \\ xy^3 & = & -1 \end{array}\right\} \Rightarrow \frac{x^2}{y^2} = 1 \text{ and } y = -\frac{1}{x^3}.$$

Case 1: If $y = x$, then

$$x = -\frac{1}{x^3} \Leftrightarrow x^4 = -1 \Rightarrow x \notin \mathbb{R}.$$

so there are no critical points in this case.
Case 2: If $y = -x$, then

$$-x = -\frac{1}{x^3} \Leftrightarrow x^4 = 1 \Rightarrow x = \pm 1;$$

so the two critical points are

$$(x, y) = (1, -1) \text{ and } (x, y) = (-1, 1).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 2 - \frac{4}{x^3 y}, \ \frac{\partial^2 z}{\partial y^2} = 2 - \frac{4}{xy^3}, \ \frac{\partial^2 z}{\partial x \partial y} = -\frac{2}{x^2 y^2}$$

and

$$\Delta = \left(2 - \frac{4}{x^3 y}\right)\left(2 - \frac{4}{xy^3}\right) - \left(-\frac{2}{x^2 y^2}\right)^2.$$

At both critical points,

$$\Delta = 32 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = 6 > 0,$$

so f has a minimum value (of $z = 4$) at both critical points $(x, y) = (1, -1)$ and $(-1, 1)$.

10. [10 marks] Consider the cycloid with vector equation $\mathbf{r} = (at - a\sin t)\,\mathbf{i} + (a - a\cos t)\,\mathbf{j}$, for $a > 0$ and $0 \le t \le 2\pi$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = a(1-\cos t)\,\mathbf{i} + a\sin t\,\mathbf{j};$$

$$\begin{aligned}
\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{a^2(1-\cos t)^2 + a^2(-\cos t)^2} \\
&= a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\
&= a\sqrt{2 - 2\cos t} \\
&= a\sqrt{4\sin^2(t/2)} \\
&= 2a\sin(t/2), \text{ since } 0 \le t/2 \le \pi
\end{aligned}$$

(b) [4 marks] Find an arc length parametrization of the above curve, with reference point $(0,0)$, for which $t = 0$.

Solution:

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = [-4a\ \cos(u/2)]_0^t = 4a - 4a\cos(t/2)$$

So

$$\cos(t/2) = \frac{4a-s}{4a} \Leftrightarrow t = 2\cos^{-1}\left(\frac{4a-s}{4a}\right).$$

Consequently an arc length parametrization of the curve is $\mathbf{r} =$

$$a\left(2\cos^{-1}\left(\frac{4a-s}{4a}\right) - \sin\left(2\cos^{-1}\left(\frac{4a-s}{4a}\right)\right)\right)\mathbf{i} + a\left(1 - \cos\left(2\cos^{-1}\left(\frac{4a-s}{4a}\right)\right)\right)\mathbf{j}.$$

Optional Simplification: use $\cos(t/2) = \dfrac{4a-s}{4a}$, $\sin(t/2) = \dfrac{\sqrt{8as - s^2}}{4a}$.

Then

$$\sin t = \frac{(4a-s)\sqrt{8as-s^2}}{8a^2} \text{ and } \cos t = \frac{8a^2 - 8as + s^2}{8a^2}$$

and

$$\mathbf{r} = \left(2a\cos^{-1}\left(\frac{4a-s}{4a}\right) - \frac{(4a-s)\sqrt{8as-s^2}}{8a}\right)\mathbf{i} + \left(\frac{8as-s^2}{8a}\right)\mathbf{j}.$$

11. [10 marks] Find all the critical points on the following curves:

(a) [5 marks] The curve with parametric equations $x = 1 + t^2$, $y = 2 + 3t - t^3$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3 - 3t^2}{2t}.$$

So

$$\frac{dy}{dx} = 0 \Leftrightarrow t = \pm 1 \text{ and } \frac{dy}{dx} \text{ is undefined } \Leftrightarrow t = 0.$$

So the three critical points are

$$(x, y) = (1, 2),\ (2, 4) \text{ or } (2, 0).$$

(b) [5 marks] The logarithmic spiral with polar equation $r = e^\theta$, for $0 \le \theta \le 2\pi$.

Solution: the parametric equations are $x = e^\theta \cos\theta;\ y = e^\theta \sin\theta$, so

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{e^\theta \sin\theta + e^\theta \cos\theta}{e^\theta \cos\theta - e^\theta \sin\theta} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}.$$

Then for $0 \le \theta \le 2\pi$,

$$\frac{dy}{dx} = 0 \Leftrightarrow \sin\theta + \cos\theta = 0 \Leftrightarrow \tan\theta = -1 \Leftrightarrow \theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

and

$$\frac{dy}{dx} \text{ is undefined } \Leftrightarrow \cos\theta - \sin\theta = 0 \Leftrightarrow \tan\theta = 1 \Leftrightarrow \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}.$$

Note: θ must be in radians! The four critical points are

$$(x, y) = \left(\frac{e^{\pi/4}}{\sqrt{2}}, \frac{e^{\pi/4}}{\sqrt{2}}\right), \left(-\frac{e^{3\pi/4}}{\sqrt{2}}, \frac{e^{3\pi/4}}{\sqrt{2}}\right), \left(-\frac{e^{5\pi/4}}{\sqrt{2}}, -\frac{e^{5\pi/4}}{\sqrt{2}}\right), \left(\frac{e^{7\pi/4}}{\sqrt{2}}, -\frac{e^{7\pi/4}}{\sqrt{2}}\right).$$