University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, JUNE, 2009**
First Year - CHE, CIV, IND, LME, MEC, MMS

MAT187H1F - CALCULUS II
Exam Type: A

Comments:

1. The only practical way to do the last question is to use the series for $e^x, \ln(1+x)$ and $\tan^{-1} x$ and adjust them accordingly. Otherwise you have to do lots and lots of differentiation!

2. In Question 11 lots of students used the binomial series with $\alpha = 1/2$, and so got a series in the denominator, which is hard to handle!

3. Question 12 is easy, if you separate variables correctly, but many students couldn't!

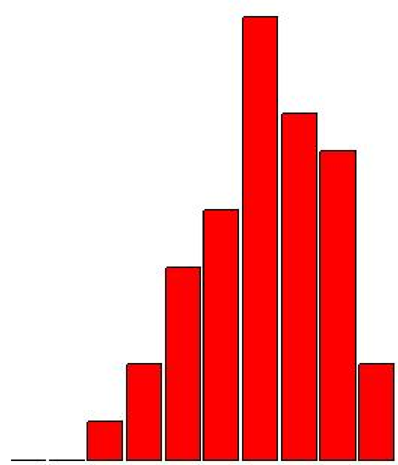
Alternate Solutions:

1. You could use the alternating series test for 7(a); or the comparison test for 7(c).

2. In 7(c), you could use the ratio test, but nobody who tried it could calculate the limit correctly! Correctly calculated, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{3}{4}$.

Breakdown of Results: 92 registered students wrote this exam. The marks ranged from 28% to 97%, and the average was 65.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.4%
A	22.8 %	80-89%	17.4%
B	19.6%	70-79%	19.6 %
C	25.0 %	60-69%	25.0%
D	14.1%	50-59%	14.1%
F	18.5 %	40-49%	10.9%
		30-39%	5.4%
		20-29%	2.2%
		10-19%	0.0%
		0-9%	0.0%



1. The position $x(t)$ of a particle at time t changes according to the differential equation

$$x''(t) + 4x'(t) + 8x(t) = 0.$$

The motion of the particle is

(a) simple harmonic motion.

(b) underdamped.

(c) critically damped.

(d) overdamped.

Solution: solve the associated quadratic.

$$r^2 + 4r + 8 = 0 \Rightarrow r = -2 \pm 2i$$

Both roots are complex, so the system is underdamped. The answer is (b).

2. What is the length of the polar curve with polar equation $r = e^{-\theta}$, for $0 \le \theta \le 1$?

(a) $\dfrac{e-1}{e}$

(b) $\sqrt{2}$

(c) $\sqrt{2}\left(\dfrac{e+1}{e}\right)$

(d) $\sqrt{2}\left(\dfrac{e-1}{e}\right)$

Solution:

$$\begin{aligned}
\int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^1 \sqrt{e^{-2\theta} + (-e^{-\theta})^2}\, d\theta \\
&= \sqrt{2}\int_0^1 e^{-\theta}\, d\theta \\
&= \sqrt{2}\left[-e^{-\theta}\right]_0^1 = \sqrt{2}(1 - 1/e)
\end{aligned}$$

The answer is (d).

3. What is the interval of convergence of the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n\, n}{3^n}(x+1)^n$?

(a) $(-4, 2)$

(b) $[-4, 2]$

(c) $[-4, 2)$

(d) $(-4, 2]$

Solution: Check convergence at endpoints. At $x = -4$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n n}{3^n}(-3)^n = \sum_{n=2}^{\infty} n,$$

which obviously diverges. At $x = 2$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n n}{3^n}(3^n) = \sum_{n=2}^{\infty} (-1)^n n,$$

which diverges by the n-th term test. The answer is (a).

4. For which values of t is the curve with parametric equations

$$x = t^2 - 12t,\ \ y = \ln(t^2+1),$$

decreasing?

(a) $0 < t < 6$

(b) $0 < t < 3$

(c) $0 < t < 2$

(d) $t < 0$ or $t > 3$

Solution:

$$\frac{dy}{dx} = \frac{\frac{2t}{t^2+1}}{2t-12} = \frac{1}{t^2+1}\frac{t}{t-6};$$

$$\frac{dy}{dx} < 0 \Leftrightarrow 0 < t < 6.$$

The answer is (a).

5. What is the area bounded by one loop of the curve with polar equation $r = \cos(3\theta)$?

(a) $\dfrac{\pi}{12}$

(b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{4}$

(d) $\dfrac{\pi}{3}$

Solution: $r = 0 \Rightarrow 3\theta = \pm\pi/2 \Rightarrow \theta = \pm\pi/6$.

$$\begin{aligned} A = 2\left(\frac{1}{2}\int_0^{\pi/6} r^2\,d\theta\right) &= \int_0^{\pi/6} \frac{1+\cos(6\theta)}{2}\,d\theta \\ &= \left[\frac{\theta}{2} + \frac{\sin(6\theta)}{12}\right]_0^{\pi/6} = \frac{\pi}{12}. \end{aligned}$$

The answer is (a).

6. What is the arc length of the curve with parametric equations

$$x = t\sin t; y = t\cos t; z = \frac{1}{3}(2t)^{3/2},$$

for $0 \le t \le 4$?

(a) 4

(b) 8

(c) 12

(d) 16

Solution:

$$\begin{aligned} &\int_0^4 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt \\ = &\int_0^4 \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + (\sqrt{2t})^2}\,dt \\ = &\int_0^4 \sqrt{1+2t+t^2}\,dt = \int_0^4 (1+t)\,dt = \left[t + \frac{t^2}{2}\right]_0^4 = 12 \end{aligned}$$

The answer is (c).

7. [12 marks; 4 for each part.] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\displaystyle\sum_{n=1}^{\infty}\left[-\arctan\left(\frac{1}{n}\right)\right]^n$ ⊗ Converges ○ Diverges

by the root test

Calculation:

$$\lim_{n\to\infty}\sqrt[n]{\left|-\arctan\left(\frac{1}{n}\right)^n\right|}=\lim_{n\to\infty}\arctan\left(\frac{1}{n}\right)=\arctan 0=0<1.$$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$ ⊗ Converges ○ Diverges

by the comparison test

Calculation:

$$n\geq 3\Rightarrow \ln n>1\Rightarrow a_n=\frac{1}{n^2\ln n}<\frac{1}{n^2}=b_n$$

The series $\sum b_n$ converges, since it is a p-series with $p=2>1$. So the series $\sum a_n$ also converges, by the comparison test.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n+2^n}{4^n+3^n}$ ⊗Converges ○ Diverges

by the limit comparison test

Calculation: Let

$$a_n=\frac{3^n+2^n}{4^n+3^n};\, b_n=\left(\frac{3}{4}\right)^n.$$

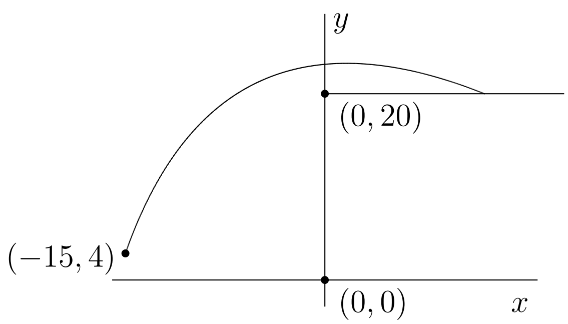
Then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{4^n}{3^n}\left(\frac{3^n+2^n}{4^n+3^n}\right)=\lim_{n\to\infty}\frac{(12)^n+8^n}{(12)^n+9^n}=\lim_{n\to\infty}\frac{1+(2/3)^n}{1+(3/4)^n}=1,$$

and the series $\sum b_n$ converges, since it is a geometric series with common ratio $r=3/4<1$. So the series $\sum a_n$ also converges, by the limit comparison test.

8. [12 marks] A fireman holds a fire hose 4 ft above the ground and aims it (towards a burning building) at an angle of 60° to the horizontal. The fireman is standing 15 ft from the building which is 20 ft high. The water leaves the hose with a speed of 40 ft/sec. Determine *exactly* where the water hits the building: on the side or on the roof. (Assume that the acceleration due to gravity is 32 ft/sec^2; ignore air resistance.)

Solution: Take $v_0 = 40$, angle of inclination $\alpha = 60°$, and let $(x, y) = (0, 0)$ be the base of the wall. So $(x_0, y_0) = (-15, 4)$.



Then

$$x = -15 + 40\,t\cos 60° = -15 + 20t$$

and

$$y = 4 + 40\,t\sin 60° - 16\,t^2 = 4 + 20\sqrt{3}\,t - 16t^2.$$

Does the water reach the roof? Yes:

$$x = 0 \Leftrightarrow t = \frac{3}{4} \Rightarrow y = 4 + 15\sqrt{3} - 9 = 15\sqrt{3} - 5 \simeq 20.98 > 20.$$

How far along the roof does the water land?

$$\begin{aligned} y = 20 \quad &\Rightarrow \quad 4 + 20\sqrt{3}\,t - 16t^2 = 0 \\ &\Rightarrow \quad t = \frac{5\sqrt{3} \pm \sqrt{11}}{8} \\ &\Rightarrow \quad t \simeq 0.668 \text{ or } t = 1.497 \end{aligned}$$

At $t \simeq 1.497,\ x \simeq -15 + 20(1.497) = 14.94$.

So the water hits the roof, about 14.94 feet from the edge.

9. [12 marks.] If x is the amount of salt disolved in a saline solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i, and r_0 is the rate at which the well-mixed solution is leaving the tank.

A 400 gallon tank initially contains 200 gallons of brine (i.e. saline solution) containing 75 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/sec, and the well-mixed brine in the tank flows out at the rate of 3 gal/sec. How much salt will the tank contain when it is full of brine?

Solution: $V = 200 + (r_i - r_o)t = 200 + 2t$. The integrating factor of the differential equation

$$\frac{dx}{dt} + \frac{3\,x}{200+2t} = 5$$

is

$$\rho = e^{\int \frac{3}{200+2t}\,dt} = e^{3\ln(200+2t)/2} = (200+2t)^{3/2}$$

and so

$$x = \frac{\int 5\,\rho\,dt}{\rho} = \frac{(200+2t)^{5/2} + C}{(200+2t)^{3/2}} = (200+2t) + \frac{C}{(200+2t)^{3/2}}.$$

Use the initial condition $t = 0, x = 75$ to find C :

$$75 = 200 + \frac{C}{200^{3/2}} \Leftrightarrow C = -125(200)^{3/2}.$$

Thus

$$x = (200+2t) - \frac{125(200)^{3/2}}{(200+2t)^{3/2}}.$$

The tank is full when $V = 400 \Leftrightarrow 200 + 2t = 400 \Leftrightarrow t = 100$. At $t = 100$,

$$x = (200+200) - \frac{125(200)^{3/2}}{(200+200)^{3/2}} = 400 - \frac{125}{2^{3/2}} \simeq 355.8.$$

So when the tank is full of brine it contains about 355.8 lb of salt.

10. [10 marks] Find and classify all the critical points of $f(x, y) = 2x^2 - 4xy + y^4 + 2$.

Solution: Let $z = 2x^2 - 4xy + y^4 + 2$.

$$\frac{\partial z}{\partial x} = 4x - 4y, \frac{\partial z}{\partial y} = -4x + 4y^3.$$

Critical points:

$$\left\{ \begin{array}{rcl} 4x - 4y & = & 0 \\ 4y^3 - 4x & = & 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} x & = & y \\ x & = & y^3 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} x & = & y \\ y^3 & = & y \end{array} \right.$$

$$y^3 = y \Rightarrow y = 0, 1 \text{ or } -1 \Rightarrow (x, y) = (0, 0), (1, 1) \text{ or } (-1, -1).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 4, \frac{\partial^2 z}{\partial y^2} = 12y^2, \frac{\partial^2 z}{\partial x \partial y} = -4; \Delta = 48y^2 - 16.$$

At $(0, 0), \Delta = -16 < 0$, so f has a saddle point at $(0, 0)$.
At $(1, 1)$ and $(-1, -1), \Delta = 32 > 0$; and

$$\frac{\partial^2 z}{\partial x^2} = 4 > 0,$$

so f has a minimum value at both $(1, 1)$ and $(-1, -1)$.

11. [10 marks] Approximate

$$\int_0^{0.5} \frac{x}{\sqrt{1+x^6}}\,dx$$

to within 10^{-6}, and explain why your approximation is correct to within 10^{-6}.

Solution: use the binomial series.

$$\begin{aligned}
& \int_0^{0.5} x\,(1+x^6)^{-1/2}\,dx \\
= & \int_0^{0.5} x\left(1-\frac{1}{2}x^6+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(x^6)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(x^6)^3+\cdots\right)dx \\
= & \int_0^{0.5} x\left(1-\frac{1}{2}x^6+\frac{3}{8}x^{12}-\frac{5}{16}x^{18}+\cdots\right)dx \\
= & \int_0^{0.5}\left(x-\frac{1}{2}x^7+\frac{3}{8}x^{13}-\frac{5}{16}x^{19}+\cdots\right)dx \\
= & \left[\frac{1}{2}x^2-\frac{1}{16}x^8+\frac{3}{112}x^{14}-\frac{5}{320}x^{20}+\cdots\right]_0^{0.5} \\
= & \ 0.125-0.00024414+0.000001634-1.4\times10^{-8}+\ldots \\
= & \ 0.124757494\ldots
\end{aligned}$$

which is correct to within $1.4\times10^{-8}<10^{-6}$, by the alternating series remainder term.

12. [10 marks] Solve for y as a function of x if $y = 0$ when $x = 0$ and

$$\frac{dy}{dx} = 2x\, e^{x^2 - 3y}.$$

Solution: separate variables.

$$\begin{aligned}
\frac{dy}{dx} = 2x\, e^{x^2-3y} \quad &\Leftrightarrow \quad \int e^{3y}\, dy = \int 2xe^{x^2}\, dx \\
&\Leftrightarrow \quad \frac{1}{3}e^{3y} = e^{x^2} + c \\
&\Leftrightarrow \quad e^{3y} = 3e^{x^2} + C \\
&\Leftrightarrow \quad 3y = \ln\left(3e^{x^2} + C\right) \\
&\Leftrightarrow \quad y = \frac{1}{3}\ln\left(3e^{x^2} + C\right)
\end{aligned}$$

To find C, use initial conditions: $(x, y) = (0, 0)$.

$$0 = \frac{1}{3}\ln\left(3e^0 + C\right) \Leftrightarrow C = -2$$

So

$$y = \frac{1}{3}\ln\left(3e^{x^2} - 2\right).$$

13. [10 marks] Find the fifth degree Taylor polynomial of

$$f(x) = e^{-x^2} + \ln(1+3x) - \tan^{-1}(x^2)$$

at $a = 0$.

Solution: use the fact that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \text{ for all } x.$$

Then

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots.$$

Similarly,

$$\begin{aligned} \ln(1+3x) &= 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \frac{(3x)^5}{5} - \cdots \\ &= 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \frac{243}{5}x^5 - \cdots \end{aligned}$$

and

$$\tan^{-1}(x^2) = x^2 - \frac{1}{3}x^6 + \cdots$$

Thus

$$P_5(x) = 1 + 3x - \frac{13}{2}x^2 + 9x^3 - \frac{79}{4}x^4 + \frac{243}{5}x^5.$$