University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, JUNE, 2008**
First Year - CHE, CIV, IND, LME, MEC, MMS

MAT 187H1F - CALCULUS II
Exam Type: A

Comments:

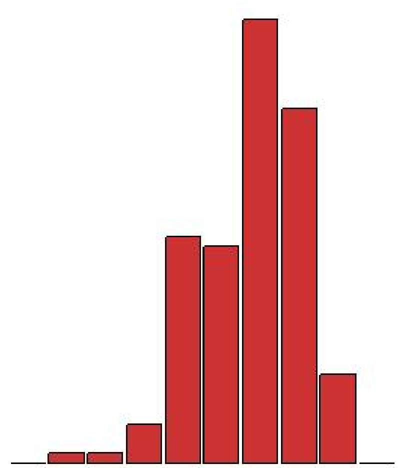
1. Questions 8, 9, 10, 12 and 13 are completely routine questions; question 12 was right out of the homework: # 27 of Section 8.4.
2. Isolating y is the hardest part of Question 11.

Alternate Solutions:

1. 7(a) also converges by the comparison test, and the ratio test, but these solutions are both much trickier
2. 7(b) also diverges by the n-th term test.
3. 7(c) also converges by the limit comparison test, with same choice of b_n; or by the integral test – but that is much harder!

Breakdown of Results: 141 students wrote this exam. The marks ranged from 15% to 84%, and the average was 61.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	6.4 %	80-89%	6.4%
B	25.5 %	70-79%	25.5%
C	31.9 %	60-69%	31.9%
D	15.6 %	50-59%	15.6%
F	20.6 %	40-49%	16.3%
		30-39%	2.8%
		20-29%	0.7%
		10-19%	0.7%
		0-9%	0.0%



1. What is the third degree Taylor polynomial of the function $f(x) = \ln(1+x)$ at $a = 0$?

   (a) $x + \frac{x^2}{2} + \frac{x^3}{3}$

   (b) $x - \frac{x^2}{2} + \frac{x^3}{3}$

   (c) $x + x^2 + x^3$

   (d) $x - x^2 + x^3$

Solution:

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots \\ \Rightarrow P_3(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} \end{aligned}$$

The answer is (b).

2. What is the area enclosed within one leaf of the three-leaved rose with polar equation $r = \sin(3\theta)$?

   (a) $\frac{1}{6}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta$

   (b) $\frac{1}{3}\int_0^{2\pi} \sin^2(3\theta)\,d\theta$

   (c) $\frac{1}{2}\int_0^{\pi/6} \sin^2(3\theta)\,d\theta$

   (d) $\frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta$

Solution:

$$\begin{aligned} \sin(3\theta) = 0 \;&\Rightarrow\; 3\theta = 0, \pm\pi, \pm 2\pi, \ldots \\ &\Rightarrow\; \theta = 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \ldots \\ \Rightarrow A \;&=\; \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta \end{aligned}$$

The answer is (d).

3. What is the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4^n}(x-2)^n$?

   (a) 4

   (b) $\frac{1}{4}$

   (c) 2

   (d) $\frac{1}{2}$

Solution:

$$\begin{aligned} a_n = \frac{\sqrt{n}}{4^n} \Rightarrow R &= \lim_{n\to\infty} \frac{a_n}{a_{n+1}} \\ &= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \frac{4^{n+1}}{4^n} \\ &= 4 \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 4 \cdot 1 = 4 \end{aligned}$$

The answer is (a).

4. If the position vector of a particle at time t is given by $\mathbf{r} = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \ln t\,\mathbf{k}$, then its speed at time $t = 1$ is

(a) 3

(b) $\sqrt{3}$

(c) 9

(d) $\frac{1}{3}$

Solution: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 2\,\mathbf{i} + 2t\,\mathbf{j} + \dfrac{1}{t}\,\mathbf{k}$.
At $t = 1$,

$$\mathbf{v} = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$$

and the speed is $|\mathbf{v}| = \sqrt{2^2 + 2^2 + 1^2} = 3$.
The answer is (a).

5. What is the slope of the tangent line to the logarithmic spiral with polar equation $r = e^\theta$ at the point $(x, y) = (-e^\pi, 0)$?

(a) 1

(b) -1

(c) π

(d) e

Solution: $x = e^\theta \cos\theta; y = e^\theta \sin\theta$.

$$\frac{dy}{dx} = \frac{e^\theta \sin\theta + e^\theta \cos\theta}{e^\theta \cos\theta - e^\theta \sin\theta} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

At $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{0 - 1}{-1 - 0} = 1$; so the answer is (a).

6. Find the length of the curve with parametric equations

$$x = 1 - 3t^2; y = 3t - t^3$$

for $0 \le t \le 1$.

(a) 1

(b) 2

(c) 4

(d) 8

Solution:

$$\begin{aligned}
&\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\
=& \int_0^1 \sqrt{(-6t)^2 + (3 - 3t^2)^2}\,dt \\
=& \int_0^1 \sqrt{9 + 18t^2 + 9t^4}\,dt \\
=& \int_0^1 (3 + 3t^2)\,dt \\
=& \left[3t + t^3\right]_0^1 = 4
\end{aligned}$$

The answer is (c).

7. [12 marks; 4 for each part.] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2^n}{n+3^n}$ ⊗ Converges ○ Diverges

by the limit comparison test

Calculation:

$$a_n = \frac{n+2^n}{n+3^n}; b_n = \left(\frac{2}{3}\right)^n \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n3^n+6^n}{n2^n+6^n} = \lim_{n\to\infty} \frac{n/2^n+1}{n/3^n+1} = 1$$

and the series $\sum b_n$ converges, since it is a geometric series with $r = 2/3 < 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \left[\ln\left(\frac{1}{n}\right)\right]^n$ ○ Converges ⊗ Diverges

by the root test

Calculation:

$$\lim_{n\to\infty} \sqrt[n]{\left|\left[\ln\left(\frac{1}{n}\right)\right]^n\right|} = \lim_{n\to\infty} |-\ln n| = \infty > 1.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ ⊗Converges ○ Diverges

by the comparison test

Calculation:

$$a_n = \ln\left(1+\frac{1}{n^2}\right); b_n = \frac{1}{n^2}. \text{ Then } a_n < b_n \left(\Leftrightarrow 1+\frac{1}{n^2} < e^{1/n^2}\right)$$

and the series $\sum b_n$ converges, since it is a p-series with $p = 2 > 1$.

8. [12 marks] The displacement, $x(t)$, of an underdamped mass-spring system satisfies

$$9x''(t) + 6x'(t) + 145x(t) = 0; x(0) = 3 \text{ and } x'(0) = 7.$$

Solve for x as a function of t and sketch its graph for $0 \leq t \leq \pi$ indicating both its pseudo period and its time-varying amplitude.

Solution: the auxiliary quadratic is $9r^2 + 6r + 145$. Solve:

$$9r^2 + 6r + 145 = 0 \Leftrightarrow r = \frac{-6 \pm \sqrt{36 - 5220}}{18} = \frac{-6 \pm 72\,i}{18} = -\frac{1}{3} \pm 4i.$$

Thus

$$x = C_1 e^{-t/3} \cos(4t) + C_2 e^{-t/3} \sin(4t).$$

To find C_1 use the initial condition $x = 3$ when $t = 0$:

$$3 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 3.$$

To find C_2 you need to find $x'(t)$:

$$x' = C_1 \left(-\frac{e^{-t/3}}{3} \cos(4t) - 4e^{-t/3} \sin(4t) \right) + C_2 \left(-\frac{e^{-t/3}}{3} \sin(4t) + 4e^{-t/3} \cos(4t) \right).$$

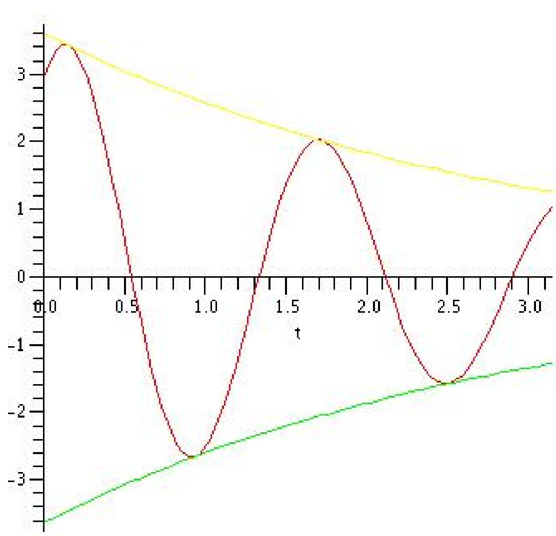
Now substitute $t = 0, x' = 7, C_1 = 3$:

$$7 = 3\left(-\frac{1}{3} - 0 \right) + C_2(0 + 4) \Leftrightarrow 8 = 4C_2 \Leftrightarrow C_2 = 2.$$

Thus

$$x = 3e^{-t/3} \cos(4t) + 2e^{-t/3} \sin(4t).$$

Graph:



The **pseudo period** is

$$\frac{2\pi}{4} = \frac{\pi}{2};$$

time-varying amplitude is

$$\sqrt{3^2 + 2^2} e^{-t/3} = \sqrt{13}\, e^{-t/3}.$$

9.[12 marks: 6 for each part.]

(a) Write down the first three non-zero terms of the Maclaurin series for each of $f(x) = x^2 \sin x$ and $g(x) = \int_0^x f(t)\,dt$.

Solution:

$$
\begin{aligned}
f(x) &= x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= x^3 - \frac{x^5}{6} + \frac{x^7}{120} - \cdots
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= \int_0^x \left(t^3 - \frac{t^5}{6} + \frac{t^7}{120} - \cdots\right) dt \\
&= \left[\frac{t^4}{4} - \frac{t^6}{36} + \frac{t^8}{960} - \cdots\right]_0^x \\
&= \frac{x^4}{4} - \frac{x^6}{36} + \frac{x^8}{960} - \cdots
\end{aligned}
$$

(b) Approximate $\int_0^{0.5} (1+x^4)^{3/2}\,dx$ to within 10^{-6}, and explain why your approximation is correct to within 10^{-6}.

Solution: use the binomial theorem.

$$
\begin{aligned}
(1+x^4)^{3/2}dx &= 1 + \frac{3}{2}x^4 + \frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2!}(x^4)^2 + \frac{\frac{3}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(x^4)^3 + \cdots \\
&= 1 + \frac{3}{2}x^4 + \frac{3}{8}x^8 - \frac{1}{16}x^{12} + \cdots \\
\Rightarrow \int_0^{0.5} (1+x^4)^{3/2}\,dx &= \left[x + \frac{3}{10}x^5 + \frac{1}{24}x^9 - \frac{1}{208}x^{13} + \cdots\right]_0^{0.5} \\
&= 0.5 + 0.009375 + 0.000081380208331\ldots \\
&= 0.50945638\ldots
\end{aligned}
$$

which is correct to within $\frac{(0.5)^{13}}{208} = 0.0000005868765\cdots < 10^{-6}$ by the alternating series remainder term.

10. [10 marks] Find and classify all the critical points of $f(x,y) = x + y + x^2y + xy^2$.

Solution: Let $z = x + y + x^2y + xy^2$.

$$\frac{\partial z}{\partial x} = 1 + 2xy + y^2, \frac{\partial z}{\partial y} = 1 + x^2 + 2xy.$$

Critical points:

$$\left\{ \begin{array}{rcl} 2xy + y^2 & = & -1 \\ x^2 + 2xy & = & -1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} y^2 - x^2 & = & 0 \\ x^2 + 2xy & = & -1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} y & = & \pm x \\ x^2 + 2xy & = & -1 \end{array} \right.$$

$$y = x \Rightarrow 3x^2 = -1, \text{ which has no real solution.}$$

$$y = -x \Rightarrow x^2 = 1 \Rightarrow (x,y) = (1,-1) \text{ or } (-1,1).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 2y, \frac{\partial^2 z}{\partial y^2} = 2x, \frac{\partial^2 z}{\partial x \partial y} = 2x + 2y; \Delta = 4xy - 4(x+y)^2.$$

At $(1,-1), \Delta = -4 < 0$, and at $(-1,1), \Delta = -4 < 0$; so both critical points

$$(1,-1,0) \text{ and } (-1,1,0)$$

are saddle points.

11. [10 marks] Solve the initial value problem

$$\text{DE: } \frac{dy}{dx} = \frac{y-y^2}{x^2+1};\ \text{IC: } y = 2 \text{ if } x = 0$$

for y as a function of x.

Solution: separate variables.

$$\begin{aligned}
\frac{dy}{dx} = \frac{y-y^2}{x^2+1} \quad &\Leftrightarrow \quad \int \frac{1}{y-y^2}\,dy = \int \frac{1}{x^2+1}\,dx \\
&\Leftrightarrow \quad \int \frac{1}{y(1-y)}\,dy = \tan^{-1}x + C \\
\text{(by partial fractions)} \quad &\Leftrightarrow \quad \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \tan^{-1}x + C \\
&\Leftrightarrow \quad \ln|y| - \ln|1-y| = \tan^{-1}x + C \\
&\Leftrightarrow \quad \ln\left|\frac{y}{1-y}\right| = \tan^{-1}x + C
\end{aligned}$$

To find C, use IC: $\ln\left|\frac{2}{1-2}\right| = 0 + C \Leftrightarrow C = \ln 2$. So

$$\begin{aligned}
\ln\left|\frac{y}{1-y}\right| &= \tan^{-1}x + \ln 2 \\
\Rightarrow \left|\frac{y}{1-y}\right| &= e^{\tan^{-1}x+\ln 2} = 2e^{tan^{-1}x} \\
\Rightarrow \frac{y}{1-y} &= -2e^{\tan^{-1}x},\ \text{by IC} \\
\Rightarrow \frac{1-y}{y} &= -\frac{1}{2}e^{-\tan^{-1}x} \\
\Rightarrow \frac{1}{y} - 1 &= -\frac{1}{2}e^{-\tan^{-1}x} \\
\Rightarrow \frac{1}{y} &= 1 - \frac{1}{2}e^{-\tan^{-1}x} \\
\Rightarrow y &= \frac{1}{1-\frac{1}{2}e^{-\tan^{-1}x}} \text{ or } \frac{2}{2-e^{-\tan^{-1}x}}
\end{aligned}$$

12. [10 marks] If x is the amount of salt disolved in a saline solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i, and r_0 is the rate at which the well-mixed solution is leaving the tank.

A 400 gallon tank initially contains 100 gallons of brine (i.e. saline solution) containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/sec, and the well-mixed brine in the tank flows out at the rate of 3 gal/sec. How much salt will the tank contain when it is full of brine?

Solution: $V = 100 + (r_i - r_o)t = 100 + 2t$. The integrating factor of the differential equation

$$\frac{dx}{dt} + \frac{3}{100+2t}x = 5$$

is

$$\rho = e^{\int \frac{3}{100+2t}\,dt} = e^{3\ln(100+2t)/2} = (100+2t)^{3/2}$$

and so

$$x = \frac{\int 5\,\rho\,dt}{\rho} = \frac{(100+2t)^{5/2} + C}{(100+2t)^{3/2}} = (100+2t) + \frac{C}{(100+2t)^{3/2}}.$$

Use the initial condition $t = 0, x = 50$ to find C :

$$50 = 100 + \frac{C}{100^{3/2}} \Leftrightarrow C = -50\,000.$$

Thus

$$x = (100+2t) - \frac{50\,000}{(100+2t)^{3/2}}.$$

The tank is full when $V = 400 \Leftrightarrow 100 + 2t = 400 \Leftrightarrow t = 150$. At $t = 150$,

$$x = (100+300) - \frac{50\,000}{(100+300)^{3/2}} = 400 - \frac{25}{4} = 393.75.$$

So when the tank is full of brine it contains about 393.75 lb of salt.

13. [10 marks] Use power series to find the Taylor series of

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

at $a = 0$; then answer the following questions about it, for 2 marks each:

(a) What is its interval of convergence?

(b) How does it compare with the Taylor series of $\cos x$?

Solution: use the fact that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \text{ for all } x.$$

Then

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots.$$

So

$$\begin{aligned}
\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
&= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\right) \\
&= \frac{1}{2}\left(2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \cdots\right) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots, \text{ for all } x
\end{aligned}$$

(a) The interval of convergence of the series for $\cosh x$ is $(-\infty, \infty) = \mathbb{R}$, since the series for e^x converges for all x.

(b) The Taylor series of $\cos x$ at $a = 0$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

The two series are almost identical; the series for $\cos x$ is the alternating version of the series of $\cosh x$.