MAT187H1F - Calculus II - Spring 2017

Solutions to Term Test 2 - June 14, 2017

Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator; formula sheet.

Comments:

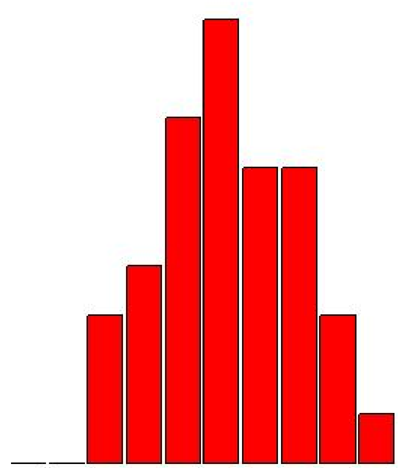
1. As with the first test, the results on this test were mediocre. And as with the first test, to raise the overall average slightly, I will count this test out of the top mark, which was 73. This effectively raises the average on this test to 62.8%, from 57%.

2. Every question was done perfectly at least once, except for Question 6, which had a top mark of only 5 out of 10.

3. The last question was poorly done, as a group. Though 6 students got it perfect, 12 students had zero on it!

Breakdown of Results: 38 students wrote this test. The marks ranged from 25% to 91%, and the average was 57%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.6%
A	10.5%	80-89%	7.9%
B	15.8%	70-79%	15.8%
C	15.8%	60-69%	15.8%
D	23.7%	50-59%	23.7%
F	34.2%	40-49%	18.4%
		30-39%	10.5%
		20-29%	5.3%
		10-19%	0.0%
		0-9%	0.0%



PART I : No explanation is necessary. Answer all of the following short-answer questions by putting your answer in the appropriate blank or by circling your choice(s).

1. [5 marks; 1 mark for each part. Avg: 3/5]

(a) $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{3}{5}\right)^k =$ 5/8

$$\sum_{k=0}^{\infty}\left(-\frac{3}{5}\right)^k = \frac{1}{1-(-3/5)} = \frac{5}{5+3} = \frac{5}{8}$$

(b) $\displaystyle\sum_{k=0}^{\infty}\frac{2}{(k+1)(k+3)} =$ 3/2

$$\sum_{k=0}^{\infty}\frac{2}{(k+1)(k+3)} = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+3}\right) = \frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\cdots = \frac{3}{2}$$

(c) The infinite series $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}}$

(i) converges absolutely $\boxed{\text{(ii)}}$ converges conditionally (iii) diverges

(d) The infinite series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{2k+1}$

(i) converges absolutely (ii) converges conditionally $\boxed{\text{(iii)}}$ diverges

(e) The infinite series $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k^2+1}$

$\boxed{\text{(i)}}$ converges absolutely (ii) converges conditionally (iii) diverges

2. [10 marks; 2 marks for each part. Avg: 4/10] Consider the power series $f(x) = \sum_{k=1}^{\infty} k\, x^{k-1}$.

(a) What is the radius of convergence of $f(x)$? $\underline{R = 1}$

$$a_k = k;\ \lim_{k\to\infty} \frac{a_k}{a_{k+1}} = \lim_{k\to\infty} \frac{k}{k+1} = 1.$$

(b) What is the open interval of convergence of $f(x)$? Answer: $\underline{(-1, 1)}$

For parts (c) and (d), assume x is in the open interval of convergence of $f(x)$.

(c) What is the power series for $\int_0^x f(t)\, dt$, in sigma notation? $\int_0^x f(t)\, dt = \underline{\sum_{k=1}^{\infty} x^k}$.

(d) Find a formula for $\int_0^x f(t)\, dt$, not in terms of a power series. $\int_0^x f(t)\, dt = \underline{\frac{x}{1-x}}$.

$$\sum_{k=1}^{\infty} x^k = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

(e) What is the exact value of $f(1/2)$? $f(1/2) = \underline{4}$

$$f(x) = \frac{d\left(\frac{x}{1-x}\right)}{dx} = \frac{1}{(1-x)^2} \Rightarrow f(1/2) = \frac{1}{(1-1/2)^2} = 4$$

PART II : Present **complete** solutions to the following questions in the space provided.

3. Avg: 11.1/15

3. [15 marks] Consider the curve with parametric equations $x = t^3 - 3t,\ y = t^4 - 4t^3$.

   (a) [3 marks] Find the first derivative $\dfrac{dy}{dx}$ in terms of t.

   Solution:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3 - 12t^2}{3t^2 - 3} = \frac{4t^2(t-3)}{3(t^2-1)}.$$

   (b) [7 marks] Find the coordinates, (x, y), of all the critical[1] points on the curve.

   Solution: need to use your derivative from part (a): $\dfrac{dy}{dx} = 0$ for $t = 0$ or $t = 3$; and $\dfrac{dy}{dx}$ is undefined for $t = \pm 1$. Thus the four critical points are

   1. $(x, y) = (0, 0)$, for $t = 0$.
   2. $(x, y) = (18, -27)$, for $t = 3$.
   3. $(x, y) = (2, 5)$, for $t = -1$.
   4. $(x, y) = (-2, -3)$, for $t = 1$.

[1] Recall: a critical point on a curve is a point where the derivative is zero or undefined.

(c) [5 marks] It was shown in class that the second derivative in parametric form is given by

$$\frac{d^2y}{dx^2} = \frac{\frac{d\,y'}{dt}}{\frac{dx}{dt}},$$

where $y' = \dfrac{dy}{dx}$. Find the second derivative $\dfrac{d^2y}{dx^2}$ in terms of t, and use it to determine if any of the critical points you found in part (b) is a maximum or minimum point.

Solution: you need to differentiate your answer from part (a) with respect to t :

$$\frac{d\,y'}{dt} = \frac{d\left(\frac{4t^3-12t^2}{3t^2-3}\right)}{dt} = \frac{(12t^2-24t)(3t^2-3)-(4t^3-12t^2)6t}{(3t^2-3)^2} = \frac{4}{3}\frac{t(t^3-3t+6)}{(t^2-1)^2}$$

Then

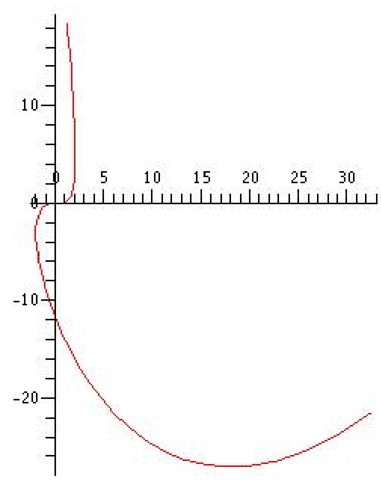
$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{4}{9}\frac{t(t^3-3t+6)}{(t^2-1)^3}.$$

Now for $t = \pm 1$ the second derivative is undefined, and for $t = 0$ the second derivative is zero. So the only useful information from the second derivative is at $t = 3$ for which

$$\frac{d^2y}{dx^2} = \frac{96}{192} = \frac{1}{2} > 0;$$

so there is a minimum point on the curve at $(x, y) = (18, -27)$.

For interest here is the graph of the curve:



4. [10 marks. Avg: 6.3/10] Let $f(x) = \cos(2x)\,\sin(x^2)$.

(a) Find the first three non-zero terms in the Maclaurin series of

(*i*) [2 marks] $\cos(2x)$

Solution:

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots = 1 - 2x^2 + \frac{2x^4}{3} - \cdots$$

(*ii*) [2 marks] $\sin(x^2)$

Solution:

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \cdots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots$$

(*iii*) [3 marks] $f(x)$

Solution:

$$\begin{aligned} f(x) = \cos(2x)\,\sin(x^2) &= \left(1 - 2x^2 + \frac{2x^4}{3} - \cdots\right)\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots\right) \\ &= x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} + \cdots - 2x^4 + \frac{x^8}{3} - \frac{x^{12}}{60} + \cdots + \frac{2x^6}{3} - \frac{x^{10}}{9} + \frac{x^{14}}{180} + \cdots \\ &= x^2 - 2x^4 + \frac{x^6}{2} + \cdots \end{aligned}$$

(b) [3 marks] What is the value of $f^{(6)}(0)$?

Solution: since the power series for $f(x)$ is the Maclaurin series for $f(x)$ we have

$$\frac{1}{2} = \frac{f^{(6)}(0)}{6!} \Leftrightarrow f^{(6)}(0) = 360.$$

5. [10 marks. Avg: 7.1/10] Approximate the value of

$$\int_0^{1/2} \frac{x^2}{\sqrt{1+x^4}}\,dx$$

correct to within 10^{-4}. Make sure to explain why your approximation *is* correct to within 10^{-4}.

Solution: use binomial series.

$$\begin{aligned}
\frac{x^2}{\sqrt{1+x^4}} &= x^2(1+x^4)^{-1/2} \\
&= x^2\left(1+(-1/2)x^4+\frac{(-1/2)(-3/2)}{2!}(x^4)^2+\frac{(-1/2)(-3/2)(-5/2)}{3!}(x^4)^3+\cdots\right) \\
&= x^2\left(1-\frac{x^4}{2}+\frac{3\,x^8}{8}-\frac{5\,x^{12}}{16}+\cdots\right) \\
&= x^2-\frac{x^6}{2}+\frac{3\,x^{10}}{8}-\frac{5\,x^{14}}{16}+\cdots \\
\Rightarrow \int_0^{1/2} \frac{x^2}{\sqrt{1+x^4}}\,dx &= \int_0^{1/2}\left(x^2-\frac{x^6}{2}+\frac{3\,x^{10}}{8}-\frac{5\,x^{14}}{16}+\cdots\right)dx \\
&= \left[\frac{x^3}{3}-\frac{x^7}{14}+\frac{3\,x^{11}}{88}-\frac{x^{15}}{48}+\cdots\right]_0^{1/2} \\
&= \frac{1}{24}-\frac{1}{1792}+\frac{3}{180224}-\frac{1}{1572864}+\cdots \\
&= \underbrace{0.0416666667-0.000558035}\cdots+0.000016645\cdots-0.000000635\cdots+\cdots \\
&= 0.04110862,\ \text{correct to within } 0.000016645 < 10^{-4},
\end{aligned}$$

by the alternating series test remainder formula.

6. [10 marks. Avg: 2.1/10] Consider the power series $f(x) = \sum_{n=1}^{\infty} (n+1)\, n\, x^n$.

(a) [3 marks] What is the interval of convergence of $f(x)$?

Solution: let the radius of convergence be R. We have

$$a_n = (n+1)\, n; \;\; R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)n}{(n+2)(n+1)} = \lim_{n\to\infty} \frac{n}{n+2} = 1.$$

So the open interval of convergence is $(-1, 1)$. This is also the interval of convergence since neither series,

$$\sum_{n=1}^{\infty} (n+1)\, n \quad \text{nor} \quad \sum_{n=1}^{\infty} (n+1)\, n\, (-1)^n,$$

converges, by the n-th term test for divergence:

$$\lim_{n\to\infty} a_n = \infty \neq 0.$$

(b) [7 marks] Find a formula for $f(x)$ not in terms of power series.

Solution:

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} (n+1)\, n\, x^n \\
&= x \sum_{n=1}^{\infty} (n+1)\, n\, x^{n-1} \\
&= x \sum_{n=1}^{\infty} \frac{d^2\, x^{n+1}}{dx^2} \\
&= x \frac{d^2}{dx^2} \left(\sum_{n=1}^{\infty} x^{n+1} \right), \text{ as long as } |x| < 1, \\
&= x \frac{d^2}{dx^2} \left(\frac{1}{1-x} - 1 - x \right) \\
&= x \frac{d}{dx} \left(\frac{1}{(1-x)^2} - 1 \right) \\
&= x \left(\frac{2}{(1-x)^3} \right) \\
&= \frac{2x}{(1-x)^3}
\end{aligned}$$

7. [10 marks. Avg: 7.5/10] For the given convergent infinite series determine *at least* how many terms of the series must be added up to approximate the sum of the series correctly to within 10^{-6}.

   (a) [4 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^6}$.

   Solution: use the alternating series remainder term.

$$|R_n| < a_{n+1} = \frac{1}{(n+1)^6} < 10^{-6} \Rightarrow (n+1)^6 > 10^6 \Rightarrow n+1 > 10 \Rightarrow n > 9.$$

   So you have to add at least 10 terms.

   (b) [6 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^6}$.

   Solution: use the integral test remainder term.

$$R_n < \int_n^{\infty} \frac{1}{x^6}\,dx = \lim_{b\to\infty} \left[-\frac{1}{5x^5}\right]_n^b = \frac{1}{5n^5} < 10^{-6} \Rightarrow 200000 < n^5 \Rightarrow n > 10 \cdot 2^{1/5} \approx 11.5.$$

   So you have to add at least 12 terms.

8. [10 marks. Avg: 4.6/10] By making use of appropriate series, find the following limits:

(a) [5 marks] $\displaystyle\lim_{x\to 0} \frac{x^3 + x^9 - \tan^{-1}(x^3)}{x\,\sin(x^8)}$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^3 + x^9 - \tan^{-1}(x^3)}{x\,\sin(x^8)} &= \lim_{x\to 0} \frac{x^3 + x^9 - \left(x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \cdots\right)}{x\left(x^8 - \frac{x^{24}}{3!} + \frac{x^{40}}{5!} + \cdots\right)} \\
&= \lim_{x\to 0} \frac{\frac{4x^9}{3} - \frac{x^{15}}{5} + \cdots}{x^9 - \frac{x^{25}}{3!} + \frac{x^{41}}{5!} + \cdots} \\
&= \lim_{x\to 0} \frac{\frac{4}{3} - \frac{x^6}{5} + \cdots}{1 - \frac{x^{16}}{3!} + \frac{x^{32}}{5!} + \cdots} \\
&= \frac{4}{3}
\end{aligned}$$

(b) [5 marks] $\displaystyle\lim_{x\to\infty} \frac{e^{5/x} - 1}{\ln(1 + 2/x)}$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^{5/x} - 1}{\ln(1 + 2/x)} &= \lim_{x\to\infty} \frac{\left(1 + 5/x + (5/x)^2/2! + (5/x)^3/3! + \cdots\right) - 1}{2/x - (2/x)^2/2 + (2/x)^3/3 - \cdots} \\
&= \lim_{x\to\infty} \frac{5/x + (5/x)^2/2! + (5/x)^3/3! + \cdots}{2/x - (2/x)^2/2 + (2/x)^3/3 - \cdots} \\
&= \lim_{x\to\infty} \frac{5 + (5^2/x)/2! + (5^3/x^2)/3! + \cdots}{2 - (2^2/x)/2 + (2^3/x^2)/3 - \cdots} \\
&= \frac{5}{2}
\end{aligned}$$

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