MAT187H1F - Calculus II - Spring 2015

Solutions to Term Test 2 - June 10, 2015

Time allotted: 100 minutes.

Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- The results on this test were very low. Only 39 of 89 students passed the test. Questions 1, 2, 3, 4 and 5 each had passing averages, and Question 7 *nearly* had a passing average. So far so good. But Questions 6 and 8 had averages of less than 2 out of 10.
- 2. The number of zeros was astounding: 11 on Question 3; 26 on Question 5; 39 on Question 6; 18 on Question 7; and 25 on Question 8.
- 3. Question 6 was based on homework Question 59 from Section 9.2, almost verbatim, and was an easier version of Question 6 from Problem Set 3. It should not have caught anybody by surprise.
- 4. In Question 8, parts (a) and (b) are just algebra and require no knowledge of infinite series at all.
- 5. Be that as it may, I will count this test out of 64, which increases the average to 59.8%, and reduces the number of failures from 50 to 20. It also produces a term mark of 31.4/50, approximately.
- 6. Some further comments, including some comments from the marker, can be found on page 11.

Breakdown of Results: 89 students wrote this test. The marks ranged from 12.5% to 78.75%, and the average was only 47.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	0.0%	80-89%	0.0%
В	3.4%	70-79%	3.4%
C	23.6%	60-69%	23.6%
D	16.8%	50-59%	16.8%
F	56.2%	40-49%	31.5%
		30 - 39%	11.2%
		20-29%	10.1%
		10-19%	3.4%
		0-9%	0.0%



PART I : No explanation is necessary. But part marks are possible, if your answer is incorrect. Answer all of the following short-answer questions by putting your answer in the appropriate blank or by circling your choice(s).

1. [10 marks; avg: 7.78]

(a)
$$[2 \text{ marks}] \sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^k =$$
 (infinite geometric series)

(b) [2 marks]
$$\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)} =$$

(telescoping series)

1/2

(c) [2 marks]
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2}$$

(i) converges absolutely (*ii*) converges conditionally (*iii*) diverges

(d) [2 marks]
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$$

(i) converges absolutely (ii) converges conditionally (iii) diverges

(e) [2 mark] For which values of p does the infinite series $\sum_{k=2}^{\infty} \frac{1}{k (\ln k)^p}$ converge? p > 1 (use integral test)

2. [10 marks; avg: 6.12]

(a) [6 marks] Match the sum of each infinite series in the column on the left with the appropriate number in the column on the right, by putting the appropriate letter in the appropriate blank.

A.
$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$$
 B $\sin(2)$

B.
$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}$$
 ______ 2/3
A e^{-2}

C.
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\frac{E}{D}$$

$$\frac{\ln 2}{3/2}$$

D.
$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$
 e^2
F $(8/7)^{1/3}$

F.
$$1 + \sum_{n=1}^{\infty} \frac{(1/3)(-2/3)(-5/3)\cdots(1/3-n+1)}{n!} \frac{1}{7^n}$$
 (1/7)^{1/3}

(b) [2 mark] What is the radius of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{k^5}{3^k} x^k?$$

 $R = __3$

(c) [2 mark] Write the function

$$f(x) = \frac{1}{(1-x)^2},$$

for |x| < 1, as a Maclaurin series, in sigma notation.

$$f(x) = \sum_{\substack{n=1\\(\overline{f(x)} \text{ is the derivative of } 1/(1-x))}^{\infty} nx^{n-1} \text{ or } \sum_{\substack{n=0\\n=0}}^{\infty} (n+1)x^n$$

PART II : Present **complete** solutions to the following questions in the space provided.

3. [10 marks; avg: 5.58] Recall that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$
 for all x ,
 $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$ for $|x| \le 1$.

(a) [7 marks] Find the first four non-zero terms in the Maclaurin series of $f(x) = e^{-x} \tan^{-1}(x^2)$. Solution: use series for e^{-x} and $\tan^{-1}(x^2)$ and expand the product:

$$f(x) = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right) \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \frac{x^{18}}{9} - \cdots\right)$$
$$= x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{6} + \cdots$$

(b) [3 marks] For $f(x) = e^{-x} \tan^{-1}(x^2)$, as in part (a), what is the value of $f^{(4)}(0)$?

Solution: use coefficient of x^4 in your answer from part (a) :

$$\frac{1}{2} = \frac{f^{(4)}(0)}{4!} \Rightarrow f^{(4)}(0) = 12$$

Not Recommended: if you try to differentiate f directly, four times, good luck! You will find $f^{(4)}(x) = \frac{\tan^{-1}(x^2)}{e^x} + \frac{12 - 8x}{e^x(1 + x^4)} + \frac{160x^3 - 120x^2 - 48x^4}{e^x(1 + x^4)^2} + \frac{768x^6 - 256x^7}{e^x(1 + x^4)^3} - \frac{768x^{10}}{e^x(1 + x^4)^4}$

from which it follows that $f^{(4)}(0) = 12$.

4. [10 marks; avg: 6.12] Indicate if each of the following infinite series converges or diverges, and **justify** your choice.

(a) [3 marks]
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2 + 4}}$$
 \bigotimes converges \bigcirc diverges

Solution: use the alternating series test. Let $a_k = \frac{1}{\sqrt{k^2 + 4}}$, then $\lim_{k \to \infty} \frac{1}{\sqrt{k^2 + 4}} = 0$ and $\{a_k\}$ is decreasing:

 $a_{k+1} < a_k \Leftrightarrow k^2 + 4 < k^2 + 2k + 5 \Leftrightarrow 0 < 2k + 1$, which is true.

(b) [3 marks]
$$\sum_{k=0}^{\infty} \frac{4k^2 + 6}{8k^5 + k + 10}$$
 \bigotimes converges \bigcirc diverges

Solution: use the limit comparison test with $a_k = \frac{4k^2 + 6}{8k^5 + k + 10}$, $b_k = \frac{1}{k^3}$. $\sum b_k$ converges, because it is a *p*-series with p = 3 > 1, and

$$L = \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{4k^5 + 6k^3}{8k^5 + k + 10} = \frac{1}{2}.$$

Since $0 < L < \infty$ and $\sum b_k$ converges, so does $\sum a_k$.

NB: the comparison test will work too, since

$$a_k < \frac{4k^2 + 6}{8k^5} = \frac{1}{2}\frac{1}{k^3} + \frac{3}{4}\frac{1}{k^5} < \frac{5}{4}\frac{1}{k^3}.$$

(c) [4 marks]
$$\sum_{k=1}^{\infty} \frac{(k+1)^k}{k^{k+1}}$$
 \bigcirc converges \bigotimes diverges

Solution: this diverges by the comparison test, or by the limit comparison test. For either approach, let

$$a_k = \frac{(k+1)^k}{k^{k+1}}, \ b_k = \frac{1}{k}.$$

 $\sum b_k$ diverges because it is the harmonic series. To use the **comparison test** observe that

$$a_k = \frac{(k+1)^k}{k^{k+1}} = \left(\frac{k+1}{k}\right)^k \frac{1}{k} = \left(1 + \frac{1}{k}\right)^k \frac{1}{k} > \frac{1}{k} = b_k$$

To use the **limit comparison test** observe that

$$L = \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{(k+1)^k}{k^k} = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k = e$$

Continued...

5. [10 marks; avg: 5.26] Approximate the value of

$$\int_0^{1/2} \frac{x^2}{(1+x^4)^{3/2}} \, dx$$

correct to within 10^{-4} . Make sure to explain why your approximation is correct to within 10^{-4} .

Solution: use the binomial series, and then integrate term by term.

$$\begin{split} \int_{0}^{1/2} \frac{x^{2} dx}{(1+x^{4})^{3/2}} &= \int_{0}^{1/2} x^{2} (1+x^{4})^{-3/2} dx \\ &= \int_{0}^{1/2} x^{2} \left(1 - \frac{3}{2}x^{4} + \frac{(-3/2)(-5/2)}{2!} (x^{4})^{2} + \frac{(-3/2)(-5/2)(-7/2)}{3!} (x^{4})^{3} + \cdots \right) dx \\ &= \int_{0}^{1/2} x^{2} \left(1 - \frac{3x^{4}}{2} + \frac{15x^{8}}{8} - \frac{35x^{12}}{16} + \cdots \right) dx \\ &= \int_{0}^{1/2} \left(x^{2} - \frac{3x^{6}}{2} + \frac{15x^{10}}{8} - \frac{35x^{14}}{16} + \cdots \right) dx \\ &= \left[\frac{x^{3}}{3} - \frac{3x^{7}}{14} + \frac{15x^{11}}{88} - \frac{7x^{15}}{48} + \cdots \right]_{0}^{1/2} \\ &= \frac{1}{24} - \frac{3}{1792} + \frac{15}{180224} - \frac{7}{(48)(2^{15})} + \cdots \\ &\approx 0.039992559, \text{ correct to within } \frac{15}{180224} \approx 8.3 \times 10^{-5} < 10^{-4}, \end{split}$$

by the alternating series test remainder term.

- 6. [10 marks: avg: 1.44] A patient takes 100 mg of a drug every 24 hours. The half-life of the drug in the patient's blood is 24 hours (i.e. the time it takes for half of the drug to be eliminated from the blood.)
 - (a) [3 marks] Find a recurrence relation for the sequence $\{d_n\}$ that gives the amount of drug in the blood after the *n*th dose, where $d_1 = 100$.

Solution: for $n \ge 2$ and $d_1 = 100$,

$$d_n = 100 + \frac{d_{n-1}}{2}.$$

(b) [3 marks] Use your answer from part (a) to find $\lim_{n\to\infty} d_n$.

Solution: let $L = \lim_{n \to \infty} d_n$. Take the limit of both sides of the recurrence relation in part (a) :

$$\lim_{n \to \infty} d_n = \lim_{n \to \infty} \left(100 + \frac{d_{n-1}}{2} \right) \Leftrightarrow L = 100 + \frac{L}{2} \Leftrightarrow L = 200.$$

(c) [4 marks] Find an explicit formula for d_n and then use it to find $\lim_{n\to\infty} d_n$.

Solution: for $n \ge 2$,

$$d_n = 100 + \frac{d_{n-1}}{2}$$

$$= 100 + \frac{1}{2} \left(100 + \frac{d_{n-2}}{2} \right)$$

$$= 100 + \frac{1}{2} (100) + \frac{1}{2^2} \left(100 + \frac{d_{n-3}}{2} \right)$$
...
$$= 100 + \frac{1}{2} (100) + \frac{1}{2^2} (100) + \dots + \frac{1}{2^{n-1}} (100)$$

$$= 100 \sum_{k=0}^{n-1} \left(\frac{1}{2} \right)^k$$

Then

$$\lim_{n \to \infty} d_n = 100 \lim_{n \to \infty} \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k = 100 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{100}{1 - 1/2} = 200$$

7. [10 marks: avg: 4.34] For the given convergent infinite series determine at least how many terms of the series must be added up to approximate the sum of the series correctly to within 10^{-4} .

Note: for all three parts, let S_n be the *n*-th partial sum, let S be the sum of the series, and let $R_n = |S - S_n|$ be the error term, as defined in the book. The problem is to find the least value of n such that $R_n < 10^{-4}$. (Equivalently, let $R_n = S - S_n$, and require $|R_n| < 10^{-4}$. Same difference.)

(a) [3 marks]
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$$
.

Solution: use alternating series remainder term.

$$R_n \le a_{n+1} = \frac{1}{(n+1)^4} < 10^{-4} \Rightarrow 10^4 < (n+1)^4 \Rightarrow n+1 > 10 \Rightarrow n > 9$$

So you have to add up at least 10 terms.

(b) [4 marks]
$$\sum_{k=1}^{\infty} \frac{1}{k^3}$$
.

Solution: use integral test remainder term.

$$R_n \le \int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2} < 10^{-4} \Rightarrow 5000 < n^2 \Rightarrow n > \sqrt{5000} \approx 70.7$$

So you have to add up at least 71 terms.

(c) [3 marks]
$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

Solution: use the definition of R_n and the sum of an infinite geometric series. We have

$$S = \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1-1/2} = 2: \quad S_n = \sum_{k=0}^n \left(\frac{1}{2}\right)^k = \frac{(1/2)^{n+1} - 1}{(1/2) - 1} = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

Then

$$R_n = |S - S_n| = \left|2 - 2\left(1 - \frac{1}{2^{n+1}}\right)\right| = \frac{1}{2^n} < 10^{-4} \Rightarrow 10^4 < 2^n \Rightarrow n > \frac{4}{\log 2} \approx 13.3$$

So you must add up at least 14 terms.

Alternate Solution: use integral test remainder term.

$$R_n \le \int_n^\infty 2^{-x} \, dx = \frac{1}{2^n \ln 2} < 10^{-4} \Rightarrow 10^4 < 2^n \ln 2 \Rightarrow n > 13.8$$

Continued...

8. [10 marks: avg: 1.71] Let F_n be the *n*th Fibonacci number; $F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n, n \ge 1$. Let f(x) be the power series

$$f(x) = \sum_{n=1}^{\infty} F_n x^{n-1} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$$

(a) [3 marks] Verify that $f(x) = 1/(1 - x - x^2)$ by showing that $f(x) - xf(x) - x^2f(x) = 1$.

Solution: add up the three series, term by term.

OR:
$$f(x) - xf(x) - x^2 f(x) = \sum_{n=1}^{\infty} F_n x^{n-1} - \sum_{n=1}^{\infty} F_n x^n - \sum_{n=1}^{\infty} F_n x^{n+1}$$

 $= 1 + x + \sum_{n=1}^{\infty} F_{n+2} x^{n+1} - x - \sum_{n=1}^{\infty} F_{n+1} x^{n+1} - \sum_{n=1}^{\infty} F_n x^{n+1}$
 $= 1 + \sum_{n=1}^{\infty} (F_{n+2} - F_{n+1} - F_n) x^{n+1} = 1 + 0 = 1$

(b) [3 marks] Verify that also
$$f(x) = \left(\frac{1}{\sqrt{5}}\right) \frac{\frac{1+\sqrt{5}}{2}}{\left(1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right)} - \left(\frac{1}{\sqrt{5}}\right) \frac{\frac{1-\sqrt{5}}{2}}{\left(1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right)}$$

Solution: this is the partial decomposition of f(x), but you don't have to *find* it; just check it. Its high school algebra! Show right side equals left side.

$$RS = \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2}}{\left(1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right)} - \frac{\frac{1-\sqrt{5}}{2}}{\left(1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right)} \right)$$
$$= \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2} \left(1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right) - \frac{1-\sqrt{5}}{2} \left(1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right)}{\left(1 - \left(\frac{1+\sqrt{5}}{2}\right)x\right) \left(1 - \left(\frac{1-\sqrt{5}}{2}\right)x\right)} \right)$$
$$= \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2} - \left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right)x - \frac{1-\sqrt{5}}{2} + \left(\frac{1-\sqrt{5}}{2}\right) \left(\frac{1+\sqrt{5}}{2}\right)x}{1 - \left(\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}\right)x + \left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right)x^2} \right)$$
$$= \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}}{1 - x + \left(\frac{-4}{4}\right)x^2} \right)$$
$$= \frac{1}{1 - x - x^2} = LS, \text{ by part (a)}$$

Continued...

(c) [4 marks] Find a formula for the nth Fibonacci number.

Solution: by part (b), f(x) can be written as the difference of two infinite geometric series.

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} F_n x^{n-1} \\ &= \left(\frac{1}{\sqrt{5}}\right) \frac{\frac{1+\sqrt{5}}{2}}{\left(1-\left(\frac{1+\sqrt{5}}{2}\right)x\right)} - \left(\frac{1}{\sqrt{5}}\right) \frac{\frac{1-\sqrt{5}}{2}}{\left(1-\left(\frac{1-\sqrt{5}}{2}\right)x\right)} \\ &= \left(\frac{1}{\sqrt{5}}\right) \left(\frac{1+\sqrt{5}}{2}\right) \sum_{n=1}^{\infty} \left(\left(\frac{1+\sqrt{5}}{2}\right)x\right)^{n-1} - \left(\frac{1}{\sqrt{5}}\right) \left(\frac{1-\sqrt{5}}{2}\right) \sum_{n=1}^{\infty} \left(\left(\frac{1-\sqrt{5}}{2}\right)x\right)^{n-1} \\ &= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) x^{n-1} \end{aligned}$$

Compare coefficients to conclude:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Further comments about the tests:

- 1. In Question 2(a), a surprising number of people filled in **all** the blanks.
- 2. In Question 3(b), very few students made the connection that $a_4 = \frac{f^{(4)}(0)}{4!}$.
- 3. Part (c) was the hardest part of Question 4, because neither the ratio nor the root test will work. It requires the comparison test or the limit comparison test.
- 4. The results for Question 5 exhibited a very bimodal distribution. If students knew how to start it, they usually did very well.
- 5. To quote the marker, "Question 6 was a total train wreck. That is very surprising. I really tried to give them points. Most people didn't seem to know what a 'recurrence relation' is. They either wrote gibberish or they wrote an explicit formula that came out of nowhere. Since the question explicitly asks for a recurrence relation, I couldn't give points for these other formulas. There were some weird guesses for limits, too. Barely anyone really derived the formula in part (c); they just stated memorized formulas (I still gave them marks). Not a single student stated $n \ge 2$ in part (a) which was worth a mark."
- 6. In Question 7, the most common mistake was that students tried to use the alternating series test remainder term for parts (b) and (c); but it doesn't apply.
- 7. To quote the marker, "Almost nobody finished Question 8(b). A lot of people did some algebra, but most of it led nowhere and didn't even get close to the solution. Those who wrote something usually just produced a lot of 'gibberish' and it wasn't clear where they started or where it ended. As long as people showed some effort to get to a solution (e.g. if they at least tried to get a common denominator), I gave them 1 of 3 marks, even if it was just one or two lines. There were two or three students who solved this part. In particular, one of them had a solution that was very neat."
- 8. To quote the marker, "Question 8(c) indeed caused problems. I changed the marking scheme so that just mentioning the words 'geometric series' was worth 1 out of 4 marks. Literally, just saying that word gave them a point. That way, I could give two (yes, two) students a single mark for the question. Everybody else got 0 on part 8(c). Some people just calculated the first ten elements of the sequence but most people didn't actually write anything. Barely anyone even tried to use 8(b) for this part. But most likely, most people didn't even try that part."

This page is for rough work or for extra space to finish a previous problem.