University of Toronto
Solutions to **MAT187H1F TERM TEST**
of **WEDNESDAY, JUNE 4, 2014**
Duration: 100 minutes

Only aids permitted: Casio FX-991 or Sharp EL-520 calculator.

INSTRUCTIONS: Present your solutions to all of the following questions in the booklets provided. The value of each question is indicated in parantheses beside the question number. **TOTAL MARKS: 60**

Answers:

1. [8 marks] Solve the initial value problem

$$x''(t) + 5\,x'(t) + 4\,x(t) = 0; \quad x(0) = 2; \quad x'(0) = 1$$

for x as a function of t. Ans: $x = 3e^{-t} - e^{-4t}$.

2. [8 marks] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

where T is the temperature of an object at time t, in a room with constant ambient temperature A, and k is a positive constant.

Suppose a cup of coffee, initially at temperature 90 C, is placed on a table in a room with constant temperature 20 C. If the temperature of the coffee is 70 C five minutes later, when will the temperature of the coffee be 45 C? Ans: $t = 5 \ln 2.8 / \ln 1.4$ sec.

3. [10 marks] If x is the mass of solute dissolved in a solution of volume V, at time t, in a large mixing tank, then

$$\frac{d\,x}{dt} + \frac{r_o\,x}{V} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 600 L of water and 150 kg of dissolved salt. At $t = 0$, a salt water solution containing 0.1 kg salt per L is added at a rate of 15 L/min and the mixed solution is drained off at a rate of 10 L/min. How much salt is in the tank when it reaches the point of overflowing? Ans: $x = 132.4$ kg.

4. [6 marks] Find the sum of each of the following infinite series:

(a) [2 marks] $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^k$ Ans: 3/5

(b) [4 marks] $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)}$ Ans: 1/2

5. [9 marks] Determine if the following infinite series converge or diverge, and justify your choice.

   (a) [3 marks] $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\,(\ln k)^3}$ Ans: converges by the integral test

   (b) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{4^k}$ Ans: converges by the ratio test

   (c) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{4k^2-2k+6}{8k^7+k-8}$ Ans: converges by the limit comparison test

6. [8 marks] For the given infinite series use the indicated test to establish that the series converges, and then determine how many terms of the series must be added up to approximate the sum of the series correctly to within 10^{-4}.

   (a) [4 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$; alternating series test. Ans: $n \geq 10$

   (b) [4 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$; integral test. Ans: $n \geq 15$

7. [7 marks] Solve for y in terms of x if

$$\cos x\,\frac{dy}{dx} + y = \cos^3 x.$$

Ans: $y = \dfrac{\sin x + \frac{1}{2}\sin^2 x + C}{\sec x + \tan x}$

8. [4 marks] Does the infinite series

$$\sum_{k=1}^{\infty} \frac{(k+1)^k}{k^{k+1}}$$

converge or diverge? You must justify your answer to get any marks on this question.

Ans: diverges by the limit comparison test.

Solutions:

1. [8 marks] Solve the initial value problem

$$x''(t) + 5\,x'(t) + 4\,x(t) = 0; \quad x(0) = 2; \quad x'(0) = 1$$

for x as a function of t.

Solution: the auxiliary quadratic equation is

$$r^2 + 5r + 4 = 0 \Leftrightarrow r = (r+4)(r+2) = 0 \Leftrightarrow r = -1 \text{ or } r = -4.$$

So $x = Ae^{-t} + Be^{-4t}$ and

$$x'(t) = -Ae^{-t} - 4Be^{-4t}.$$

Using the initial conditions we have:

$$A + B = 2 \text{ and } -A - 4B = 1 \Rightarrow B = -1, A = 3.$$

Thus

$$x = 3e^{-t} - e^{-4t}.$$

2. [8 marks] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

where T is the temperature of an object at time t, in a room with constant ambient temperature A, and k is a positive constant.

Suppose a cup of coffee, initially at temperature 90 C, is placed on a table in a room with constant temperature 20 C. If the temperature of the coffee is 70 C five minutes later, when will the temperature of the coffee be 45 C?

Solution: we have $A = 20$, and we can assume $T \geq 20$. Measure time in minutes, and let $t = 0$ correspond to the instant the cup is first placed on the tabel. Separate variables:

$$\int \frac{dT}{T - 20} = -\int k\,dt \Leftrightarrow \ln(T - 20) = -kt + C.$$

To find C, let $t = 0$ and $T = 90$: $\ln(90 - 20) = C \Leftrightarrow C = \ln 70$.

To find k, let $t = 5$ and $T = 70$:

$$\ln(70 - 20) = -5k + \ln 70 \Leftrightarrow k = (\ln 70 - \ln 50)/5 = 1/5 \ln 1.4.$$

Now let $T = 45$ and solve for t:

$$\ln(45-20) = -t \ln 1.4/5 + \ln 70 \Leftrightarrow t \ln 1.4/5 = \ln 70 - \ln 25 = \ln 2.8 \Leftrightarrow t = 5 \ln 2.8/\ln 1.4$$

That is, it will take about 15.3 min for the coffee to cool to 45 C.

3. [10 marks] If x is the mass of solute dissolved in a solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_o x}{V} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 600 L of water and 150 kg of dissolved salt. At $t = 0$, a salt water solution containing 0.1 kg salt per L is added at a rate of 15 L/min and the mixed solution is drained off at a rate of 10 L/min. How much salt is in the tank when it reaches the point of overflowing?

Solution: take $r_i = 15, c_i = 1/10, r_o = 10$ and $V = 600 + 5t$. Then the differential equation is

$$\frac{dx}{dt} + \frac{10\,x}{600+5t} = \frac{15}{10} \Leftrightarrow \frac{dx}{dt} + \frac{2\,x}{120+t} = \frac{3}{2},$$

which has integrating factor

$$\mu = e^{\int \frac{2\,dt}{120+t}} = e^{2\ln(120+t)} = (120+t)^2.$$

Thus

$$x = \frac{1}{(120+t)^2}\int \frac{3}{2}\,(120+t)^2\,dt = \frac{1}{(120+t)^2}\left(\frac{1}{2}(120+t)^3 + C\right) = 60+\frac{t}{2}+\frac{C}{(120+t)^2}.$$

To find C let $t = 0$ and $x = 150$: $150 = 60 + C/120^2 \Leftrightarrow C = 90(120)^2$.

The tank is full when $V = 1000 \Leftrightarrow 600 + 5t = 1000 \Leftrightarrow t = 80$.

Finally, let $t = 80$ and calculate x:

$$x = 60 + 40 + \frac{90(120)^2}{(120+80)^2} = 132.4$$

So there will be 132.4 kg of salt in the tank when it reaches the point of overflowing.

4. [6 marks] Find the sum of each of the following infinite series:

   (a) [2 marks] $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{2}{3}\right)^k$

   Solution: this is an infinite geometric series with $r = -2/3$ and its sum is

$$S = \frac{1}{1-r} = \frac{1}{1+2/3} = \frac{3}{5}.$$

(b) [4 marks] $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)}$

Solution: this is a telescoping sum. We have

$$a_k = \frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3};$$

so the n-th partial sum of the series is

$$S_n = \sum_{k=0}^{n} \frac{1}{(k+2)(k+3)} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{n+2} - \frac{1}{n+3} = \frac{1}{2} - \frac{1}{n+3}.$$

Then

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+3} \right) = \frac{1}{2}.$$

5. [9 marks] Determine if the following infinite series converge or diverge, and justify your choice.

(a) [3 marks] $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\,(\ln k)^3}$ **converges** by the integral test.

Calculation: $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^3} = \int_{\ln 2}^{\infty} \frac{du}{u^3} = \lim_{b\to\infty} \left[-\frac{1}{3u^2} \right]_{\ln 2}^{b} = 0 + \frac{1}{3\ln^2 2} < \infty$

(b) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{4^k}$ **converges** by the ratio test.

Calculation: $\displaystyle\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^2}{4^{k+1}} \frac{4^k}{k^2} = \frac{1}{4} \lim_{k\to\infty} \left(\frac{k+1}{k} \right)^2 = \frac{1}{4} \cdot 1^2 = \frac{1}{4} < 1$

(c) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{4k^2 - 2k + 6}{8k^7 + k - 8}$ **converges** by the limit comparison test.

Calculation: have $a_k = \dfrac{4k^2 - 2k + 6}{8k^7 + k - 8}$; let $b_k = \dfrac{1}{k^5}$.
Then $\sum b_k$ converges, since it is a p-series with $p = 5 > 1$, and

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{4k^7 - 2k^6 + 6k^5}{8k^7 + k - 8} = \frac{4}{8} = \frac{1}{2}.$$

Since this limit is positive and finite, $\sum a_k$ converges since $\sum b_k$ does.

6. [8 marks] For the given infinite series use the indicated test to establish that the series converges, and then determine how many terms of the series must be added up to approximate the sum of the series correctly to within 10^{-4}.

(a) [4 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$ **converges** by alternating series test.

Solution: let $a_k = \dfrac{1}{k^4}$. Then $\displaystyle\lim_{k\to\infty} a_k = 0$ and $\{a_k\}$ is decreasing since

$$a_k > a_{k+1} \Leftrightarrow (k+1)^4 > k^4 \Leftrightarrow k+1 > k.$$

So the given series converges by the alternating series test. Finally

$$|R_n| < a_{n+1} < 10^{-4} \Rightarrow \frac{1}{(n+1)^4} < 10^{-4} \Rightarrow 10^4 < (n+1)^4 \Rightarrow n+1 > 10 \Rightarrow n > 9.$$

So at least 10 terms must be added to approximate the sum of the series correctly to within 10^{-4}.

(b) [4 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$ **converges** by the integral test.

Solution:

$$\int_1^{\infty} \frac{dx}{x^4} = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_1^b = 0 + \frac{1}{3} < \infty,$$

so the given series converges by the integral test. (OR you could say it converges because it is a p-series with $p = 4 > 1$.) Finally

$$R_n < \int_n^{\infty} \frac{dx}{x^4} = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_n^b = 0 + \frac{1}{3n^3} < 10^{-4} \Rightarrow 10^4 < 3n^3$$

$$\Rightarrow n^3 > \frac{10}{3}(10)^3 \Rightarrow n > 10\left(\frac{10}{3}\right)^{1/3} \approx 14.9$$

So at least 15 terms must be added to approximate the sum of the series correctly to within 10^{-4}.

7. [7 marks] Solve for y in terms of x if

$$\cos x \frac{dy}{dx} + y = \cos^3 x.$$

Solution: this can be solved by the method of Section 8.4 if you divide through by $\cos x$ first:

$$\cos x \frac{dy}{dx} + y = \cos^3 x \Rightarrow \frac{dy}{dx} + y \sec x = \cos^2 x.$$

Then the integrating factor is

$$\mu = e^{\int \sec x \, dx} = e^{\ln|\sec x + \tan x|} = |\sec x + \tan x| = \pm(\sec x + \tan x);$$

either one will do. Then the solution is

$$y = \frac{\int \mu \cos^2 x \, dx}{\mu} = \frac{\int (\sec x + \tan x)\cos^2 x \, dx}{\sec x + \tan x} = \frac{\int (\cos x + \sin x \cos x)\, dx}{\sec x + \tan x}$$

$$= \frac{\sin x + \frac{1}{2}\sin^2 x + C}{\sec x + \tan x}$$

8. [4 marks] Does the infinite series

$$\sum_{k=1}^{\infty} \frac{(k+1)^k}{k^{k+1}}$$

converge or diverge? You must justify your answer to get any marks on this question.

Solution: $a_k = \dfrac{(k+1)^k}{k^{k+1}} = \dfrac{1}{k}\left(\dfrac{k+1}{k}\right)^k$; pick $b_k = \dfrac{1}{k}$. Now

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e.$$

Since $\sum b_k$ is the harmonic series it diverges, and since

$$0 < \lim_{k\to\infty} \frac{a_k}{b_k} < \infty,$$

the original series $\sum a_k$ also diverges, by the limit comparison test.