University of Toronto
Solutions to **MAT187H1F TERM TEST**
of **THURSDAY, JUNE 6, 2013**
Duration: 100 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Present your solutions to the following questions in the booklets provided. The value of each question is 10 marks; the value of parts of questions are indicated in parantheses beside the question number. **TOTAL MARKS: 60**

Answers:

1. Solve the initial value problem

$$x''(t) + 4\,x'(t) + 4\,x(t) = 0; \quad x(0) = 2; \quad x'(0) = 1$$

for x as a function of t and sketch the graph of x for $t \geq 0$. Ans: $x = (2 + 5t)e^{-2t}$.

2. As the salt KNO_3 dissolves in methanol the number x of grams of the salt in the solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{150}\right).$$

If $x = 30$ when $t = 0$, how long will it take for an additional 100 grams of the salt to dissolve? Ans: $t = \ln 26$ sec.

3. If x is the mass of solute dissolved in a solution of volume V, at time t, in a large mixing tank, then

$$\frac{d\,x}{dt} + \frac{r_o\,x}{V} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 700 L of water and 350 kg of dissolved salt. At $t = 0$, a salt water solution containing 1 kg salt per L is added at a rate of 20 L/min and the mixed solution is drained off at a rate of 15 L/min. How much salt is in the tank when it reaches the point of overflowing? Ans: $x = 879.95$ kg.

4. For the given infinite series use the indicated test to establish that the series converges, and then determine how many terms of the series must be added up to approximate the sum of the series correct to within 10^{-3}.

   (a) [5 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$; alternating series test. Ans: $n \geq 10^6$

   (b) [5 marks] $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$; integral test. Ans: $n > e^{1000}$

5. Determine if the following infinite series converge conditionally, converge absolutely, or diverge. You must justify your answer.

   (a) [5 marks] $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)(k+2)}$ Ans: converges conditionally.

   (b) [5 marks] $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^k+1}{k!}$ Ans: converges absolutely.

6. Let $f(x) = x\, e^x$.

   (a) [5 marks] Find the 5th Maclaurin polynomial for $f(x)$.

   Ans: $P_5(x) = x + x^2 + \dfrac{x^3}{2!} + \dfrac{x^4}{3!} + \dfrac{x^5}{4!}$

   (b) [5 marks] Write down formulas for the nth Maclaurin polynomial for $f(x)$ and the Maclaurin series for $f(x)$.

   Ans: $P_n(x) = \displaystyle\sum_{k=1}^{n} \frac{x^k}{(k-1)!}$; Maclaurin series is $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}$

Solutions:

1. [10 marks] Solve the initial value problem

$$x''(t) + 4\,x'(t) + 4\,x(t) = 0; \quad x(0) = 2; \quad x'(0) = 1$$

for x as a function of t and sketch the graph of x for $t \geq 0$.

Solution: the auxiliary quadratic equation is

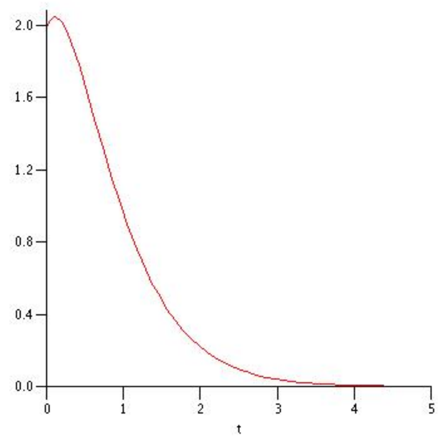
$$r^2 + 4r + 4 = 0 \Leftrightarrow r = -2.$$

So $x = Ae^{-2t} + Bte^{-2t}$ and

$$x' = -2Ae^{-2t} + Be^{-2t} - 2Bte^{-2t}.$$

Using the initial conditions we have:

$$A = 2 \text{ and } -2A + B = 1 \Rightarrow B = 5.$$



Thus $x = (2 + 5t)e^{-2t}$.

Now $x' = e^{-2t} - 10e^{-2t} = 0 \Leftrightarrow t = 1/10$, where there is a maximum point. And of course $x \to 0$ as $t \to \infty$. The graph is above on the right.

2. [10 marks] As the salt KNO_3 dissolves in methanol the number x of grams of the salt in the solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{150}\right).$$

If $x = 30$ when $t = 0$, how long will it take for an additional 100 grams of the salt to dissolve?

Solution: separate variables, and use partial fractions:

$$\int \frac{150\,dx}{x\,(150 - x)} = \int dt \Leftrightarrow \int \left(\frac{1}{x} + \frac{1}{150 - x}\right) dx = t + C \Leftrightarrow \ln x - \ln|150 - x| = t + C.$$

To find C let $t = 0$ and $x = 30$: $\ln 30 - \ln 120 = C \Leftrightarrow C = -\ln 4$. Now let $x = 130$ and solve for t:

$$\ln 130 - \ln 20 = t - \ln 4 \Rightarrow t = \ln 130 - \ln 20 + \ln 4 = \ln 26.$$

So it will take $t = \ln 26$ seconds.

3. [10 marks] If x is the mass of solute dissolved in a solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_o\, x}{V} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 700 L of water and 350 kg of dissolved salt. At $t = 0$, a salt water solution containing 1 kg salt per L is added at a rate of 20 L/min and the mixed solution is drained off at a rate of 15 L/min. How much salt is in the tank when it reaches the point of overflowing?

Solution: take $r_i = 20, c_i = 1, r_o = 15$ and $V = 700 + 5t$. Then the differential equation

$$\frac{dx}{dt} + \frac{15\, x}{700 + 5t} = 20$$

has integrating factor

$$\mu = e^{\int \frac{15\, dt}{700+5t}} = e^{3\ln(700+5t)} = (700 + 5t)^3$$

and so

$$x = \frac{1}{(700+5t)^3}\int 20\,(700+5t)^3\, dt = \frac{(700+5t)^4 + C}{(700+5t)^3} = 700 + 5t + \frac{C}{(700+5t)^3}.$$

To find C let $t = 0$ and $x = 350$: $350 = 700 + C/700^3 \Leftrightarrow C = -350(700)^3$.

The tank is full when $V = 1000 \Leftrightarrow 700 + 5t = 1000 \Leftrightarrow t = 60$.

Let $t = 60$ and calculate x:

$$x = 10000 - \frac{350(700)^3}{1000^3} = 879.95$$

4. For the given infinite series use the indicated test to establish that the series converges, and then determine how many terms of the series must be added up to approximate the sum of the series correct to within 10^{-3}.

   (a) [5 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges by alternating series test.

   Justification: let $a_k = \dfrac{1}{\sqrt{k}}$. Then $\lim\limits_{k\to\infty} a_k = 0$ and $\{a_k\}$ is decreasing since

   $$a_k > a_{k+1} \Leftrightarrow \sqrt{k+1} > \sqrt{k}.$$

   So the given series converges by the alternating series test. Finally

   $$|R_n| < a_{n+1} < 10^{-3} \Rightarrow \frac{1}{\sqrt{n+1}} < 10^{-3} \Rightarrow 10^3 < \sqrt{n+1} \Rightarrow n + 1 > 10^6.$$

(b) [5 marks] $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges by integral test.

Justification:

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = \lim_{b\to\infty}\left[-\frac{1}{u}\right]_{\ln 2}^{b} = \frac{1}{\ln 2} < \infty,$$

so the given series converges by the integral test. Finally

$$R_n < \int_n^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln n}^{\infty} \frac{du}{u^2} = \frac{1}{\ln n} < 10^{-3} \Rightarrow 10^3 < \ln n \Rightarrow n > e^{1000}.$$

5. Determine if the following infinite series converge conditionally, converge absolutely, or diverge. You must justify your answer.

(a) [5 marks] $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{(k+1)(k+2)}$ converges conditionally.

Justification: $\displaystyle\sum_{k=1}^{\infty} \frac{k}{(k+1)(k+2)}$ diverges by the Limit Comparison Test: take

$$a_k = \frac{k}{(k+1)(k+2)},\, b_k = \frac{1}{k}, \text{ then } \lim_{k\to\infty}\frac{a_k}{b_k} = 1,$$

and $\sum b_k$ diverges, since it is the harmonic series. Thus $\sum a_k$ diverges too. But

$$\sum_{k=1}^{\infty}(-1)^k\frac{k}{(k+1)(k+3)}$$

converges by the Alternating Series Test.

1. $\displaystyle\lim_{k\to\infty}\frac{k}{(k+1)(k+3)} = 0$ and
2. $\{a_k\}$ is decreasing since $a_k > a_{k+1} \Leftrightarrow$

$$\frac{k}{(k+1)(k+2)} > \frac{k+1}{(k+2)(k+3)} \Leftrightarrow k^2+3k > k^2+2k+1 \Leftrightarrow k > 1.$$

(b) [5 marks] $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^k+1}{k!}$ converges absolutely.

Justification: use the Ratio Test for absolute convergence.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{3^{k+1}+1}{(k+1)!}\,\frac{k!}{3^k+1} = \lim_{k\to\infty}\frac{1}{k+1}\,\frac{3^{k+1}+1}{3^k+1} = 0,$$

since

$$\lim_{k\to\infty}\frac{3^{k+1}+1}{3^k+1} = \lim_{k\to\infty}\frac{3+1/3^k}{1+1/3^k} = 3.$$

So the given series converges absolutely.

6. Let $f(x) = x\,e^x$.

(a) [5 marks] Find the 5th Maclaurin polynomial for $f(x)$.

Solution: use of product rule gives $f'(x) = xe^x + e^x = (1+x)e^x$. Similarly,

$f^{(2)}(x) = (2+x)e^x$, $f^{(3)}(x) = (3+x)e^x$, $f^{(4)}(x) = (4+x)e^x$, $f^{(5)}(x) = (5+x)e^x$

and so

$$f(0) = 0,\, f'(0) = 1,\, f^{(2)}(0) = 2,\, f^{(3)}(0) = 3,\, f^{(4)}(0) = 4,\, f^{(5)}(0) = 5.$$

Thus

$$P_5(x) = x + 2\left(\frac{x^2}{2}\right) + 3\left(\frac{x^3}{3!}\right) + 4\left(\frac{x^4}{4!}\right) + 5\left(\frac{x^5}{5!}\right) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \frac{x^5}{4!}.$$

(b) [5 marks] Write down formulas for the nth Maclaurin polynomial for $f(x)$ and the Maclaurin series for $f(x)$.

Solution: the patterns are $f^{(k)}(x) = (k+x)e^x$ and $f^{(k)}(0) = k$. Therefore:

$$P_n(x) = \sum_{k=1}^{n} k\left(\frac{x^k}{k!}\right) = \sum_{k=1}^{n}\frac{x^k}{(k-1)!}$$

and the Maclaurin series is simply

$$\sum_{k=1}^{\infty}\frac{x^k}{(k-1)!} = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \cdots\frac{x^k}{(k-1)!} + \cdots.$$