University of Toronto
Solutions to **MAT187H1F TERM TEST**
of **THURSDAY, JUNE 7, 2012**
Duration: 110 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Present your solutions to the following questions in the booklets provided. The value for each question is indicated in parantheses beside the question number.
TOTAL MARKS: 60

Answers:

1. [8 marks] Solve the initial value problem

$$\frac{dy}{dx} + \frac{4\,y}{x} = \frac{3}{x^2} + \frac{5}{x^6}; \quad y = 6 \text{ if } x = 1.$$

Ans: $y = \frac{1}{x} - \frac{5}{x^5} + \frac{10}{x^4}$.

2. [8 marks] Solve the initial value problem

$$x''(t) + 4\,x'(t) + 3\,x(t) = 0; \quad x(0) = 2; \quad x'(0) = -1$$

for x as a function of t. Ans: $x = -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}$.

3. [9 marks] As the salt KNO_3 dissolves in methanol the number $x(t)$ of grams of the salt in the solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right).$$

If $x = 50$ when $t = 0$, how long will it take for an additional 100 grams of the salt to dissolve? Ans: $t = 2\ln 3$ seconds.

4. [10 marks] Recall: if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 600 L of water and 150 kg of dissolved salt. At $t = 0$, a salt water solution containing 0.1 kg salt per L is added at a rate of 20 L/min and the mixed solution is drained off at a rate of 15 L/min. How much salt is in the tank when it reaches the point of overflowing? Ans: $x = 119.44$ kg.

5. [9 marks] Decide if each of the following infinite series converges or diverges. Justify your choice.

   (a) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{3k^4 - 15k^2 + 17}$ Ans: diverges by Limit Comparison Test.

   (b) [3 marks] $\displaystyle\sum_{m=1}^{\infty} \frac{\sin^2 m}{m^2}$ Ans: converges by Comparison Test.

   (c) [3 marks] $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\tan^{-1} n}{2}\right)^n$ Ans: converges by Root Test.

6. [8 marks; 4 marks for each part] Determine if the following infinite series converge conditionally, converge absolutely, or diverge.

   (a) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k+1}{k(k+3)}$ Ans: converges conditionally.

   (b) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k!)^2}{(2k-1)!}$ Ans: converges absolutely.

7. [8 marks; 4 for each part.] Find the exact sum of each of the following infinite series:

   (a) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{2}{3^k} - \frac{4}{5^{k+1}}\right)$ Ans: 2

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$ Ans: 1/2

Solutions:

1. [8 marks] Solve the initial value problem

$$\frac{dy}{dx} + \frac{4\,y}{x} = \frac{3}{x^2} + \frac{5}{x^6}; \quad y = 6 \text{ if } x = 1.$$

Solution: The integrating factor is

$$\mu = e^{\int \frac{4\,dx}{x}} = e^{4\ln|x|} = x^4$$

so

$$x^4 y = \int \left(3x^2 + \frac{5}{x^2}\right) dx = x^3 - \frac{5}{x} + C \Rightarrow y = \frac{1}{x} - \frac{5}{x^5} + \frac{C}{x^4}.$$

To find C substitute $x = 1$ and $y = 6$: $6 = 1 - 5 + C \Leftrightarrow C = 10$. Thus

$$y = \frac{1}{x} - \frac{5}{x^5} + \frac{10}{x^4}.$$

2. [8 marks] Solve the initial value problem

$$x''(t) + 4\,x'(t) + 3\,x(t) = 0; \quad x(0) = 2; \quad x'(0) = -1$$

for x as a function of t.

Solution: the auxiliary quadratic equation is $r^2 + 4r + 3 = 0 \Leftrightarrow r = -3, r = -1$. So

$$x = Ae^{-3t} + Be^{-t} \text{ and } x' = -3Ae^{-3t} - Be^{-t}.$$

Using the initial conditions we have:

$$A + B = 2 \text{ and } -3A - B = -1 \Leftrightarrow (A, B) = (-1/2, 5/2).$$

Thus

$$x = -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}.$$

3. [9 marks] As the salt KNO_3 dissolves in methanol the number $x(t)$ of grams of the salt in the solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right).$$

If $x = 50$ when $t = 0$, how long will it take for an additional 100 grams of the salt to dissolve? .

Solution: separate variables, and use partial fractions:

$$\int \frac{200\,dx}{x\,(200 - x)} = \int dt \Leftrightarrow \int \left(\frac{1}{x} + \frac{1}{200 - x}\right) dx = t + C \Leftrightarrow \ln|x| - \ln|200 - x| = t + C.$$

To find C let $t = 0$ and $x = 50$: $\ln 50 - \ln 150 = C \Leftrightarrow C = -\ln 3$. Now let $x = 150$ and solve for t:

$$\ln 150 - \ln 50 = t - \ln 3 \Rightarrow t = 2\ln 3.$$

So it will take $t = 2\ln 3$ seconds.

4. [10 marks] Recall: if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 600 L of water and 150 kg of dissolved salt. At $t = 0$, a salt water solution containing 0.1 kg salt per L is added at a rate of 20 L/min and the mixed solution is drained off at a rate of 15 L/min. How much salt is in the tank when it reaches the point of overflowing?

Solution: take $r_i = 20, c_i = 1/10, r_o = 15$ and $V = 600 + 5t$. Then the differential equation

$$\frac{d\,x}{dt} + \frac{15\,x}{600 + 5t} = 2$$

has integrating factor

$$\mu = e^{\int \frac{15\,dt}{600+5t}} = e^{3\ln(600+5t)} = (600 + 5t)^3$$

and so

$$x = \frac{1}{(600+5t)^3} \int 2\,(600+5t)^3\,dt = \frac{(600+5t)^4/10 + C}{(600+5t)^3} = \frac{600+5t}{10} + \frac{C}{(600+5t)^3}.$$

To find C let $t = 0$ and $x = 150$: $150 = 60 + C/600^3 \Leftrightarrow C = 90(600)^3$.

The tank is full when $V = 1000 \Leftrightarrow 600 + 5t = 10000 \Leftrightarrow t = 80$.

Let $t = 80$ and calculate x:

$$x = \frac{1000}{10} + \frac{90(600)^3}{1000^3} = 119.44$$

5. [9 marks] Decide if each of the following infinite series converges or diverges. Justify your choice.

   (a) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{3k^4 - 15k^2 + 17}$ diverges by Limit Comparison Test.

   Justification: let $a_k = \dfrac{k^3}{3k^4 - 15k^2 + 17}$ and $b_k = \dfrac{1}{k}$. Then

   $$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{k^4}{3k^4 - 15k^2 + 17} = \frac{1}{3},$$

   which is finite, non-zero. Since $\sum b_k$ is the harmonic series, it diverges. By the Limit Comparison Test, $\sum a_k$ diverges as well.

(b) [3 marks] $\displaystyle\sum_{m=1}^{\infty} \frac{\sin^2 m}{m^2}$ converges by Comparison Test.

Justification: let $a_m = \dfrac{\sin^2 m}{m^2}$, $b_m = \dfrac{1}{m^2}$. Since $\sin^2 m \leq 1$, we have

$$a_m \leq b_m,$$

and $\sum b_k$ converges, since it is a p-series with $p = 2 > 1$.
So by the Comparison Test, $\sum a_m$ converges as well.

(c) [3 marks] $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\tan^{-1} n}{2}\right)^n$ converges by Root Test.

Justification: use the Root Test:

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\frac{\tan^{-1} n}{2}\right)^{n/n} = \lim_{n\to\infty} \frac{\tan^{-1} n}{2} = \frac{\pi}{4} < 1;$$

so $\sum a_n$ converges.

6. [8 marks; 4 marks for each part] Determine if the following infinite series converge conditionally, converge absolutely, or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k+1}{k(k+3)}$ converges conditionally.

Justification:

$$\sum_{k=1}^{\infty} \frac{k+1}{k(k+3)}$$

diverges by the Limit Comparison Test, as in Question 5(a). But

$$\sum_{k=1}^{\infty} (-1)^k \frac{k+1}{k(k+3)}$$

converges by the Alternating Series Test. So, all in all, the given series converges conditionally.

(b) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(k!)^2}{(2k-1)!}$ converges absolutely.

Justification: use the Root Test for absolute convergence.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{((k+1)!)^2}{(k!)^2}\frac{(2k-1)!}{(2k+1)!} = \lim_{k\to\infty}\frac{(k+1)^2}{(2k+1)(2k)} = \frac{1}{4} < 1,$$

so the given series converges absolutely.

7. [8 marks; 4 for each part.] Find the exact sum of each of the following infinite series:

(a) $\displaystyle\sum_{k=0}^{\infty}\left(\frac{2}{3^k}-\frac{4}{5^{k+1}}\right) = 2$

Solution:

$$\sum_{k=0}^{\infty}\left(\frac{2}{3^k}-\frac{4}{5^{k+1}}\right) = 2\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - \frac{4}{5}\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k = \frac{2}{1-1/3} - \frac{4}{5}\frac{1}{1-1/5} = 2.$$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(n+2)(n+3)} = \frac{1}{2}$

Solution: this is a telescoping sum, once you write a_n in terms of its partial fractions:

$$\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}.$$

Then

$$\begin{aligned} S_N &= \sum_{n=0}^{N}\left(\frac{1}{n+2}-\frac{1}{n+3}\right) \\ &= \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{N+2}-\frac{1}{N+3} \\ &= \frac{1}{2}-\frac{1}{N+3}. \end{aligned}$$

Thus

$$\sum_{n=0}^{\infty}\frac{1}{(n+2)(n+3)} = \lim_{N\to\infty} S_N = \frac{1}{2}.$$