## University of Toronto Solutions to MAT187H1F TERM TEST of THURSDAY, JUNE 2, 2011

Duration: 100 minutes
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all six questions, on both sides of this page. Present your solutions in the booklets provided. The value for each question is indicated in parantheses beside the question number. TOTAL MARKS: 60

## General Comments:

1. Q1 does not require any of the methods we covered for solving DE's; instead, you simply substitute $y=a x+b / x$ into the left side of the equation, and then solve for $a$ and $b$.
2. Q3, Q4, Q5 and Q6 are four routine questions, which are almost identical to questions from last year's test. These four questions should all have been aced.
3. The three methods we learned for solving DE's were covered as follows: Q2 required separation of variables (although it could have been done by the method of the integrating factor), Q3 required the methods of Appendix L, and Q5 required the method of the integrating factor.
4. The three parts of Q4 work out easily if you use the convergence/divergence tests indicated in the solutions, but can't really be solved any other way.
5. Compared to the first test: more students got A, but also more got F.

Breakdown of Results: 100 registered students wrote this test. The marks ranged from $11.7 \%$ to $96.7 \%$, and the average was $62.0 \%$. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | $\%$ | Decade | $\%$ |
| ---: | :--- | ---: | :--- |
|  |  | $90-100 \%$ | $10 \%$ |
| A | $22 \%$ | $80-89 \%$ | $12 \%$ |
| B | $18 \%$ | $70-79 \%$ | $18 \%$ |
| C | $18 \%$ | $60-69 \%$ | $18 \%$ |
| D | $16 \%$ | $50-59 \%$ | $16 \%$ |
| F | $26 \%$ | $40-49 \%$ | $10 \%$ |
|  |  | $30-39 \%$ | $10 \%$ |
|  |  | $20-29 \%$ | $4.0 \%$ |
|  |  | $10-19 \%$ | $2.0 \%$ |
|  |  | $0-9 \%$ | $0.0 \%$ |



1. [10 marks] Find the values of $a$ and $b$ for which

$$
y=a x+\frac{b}{x}
$$

is a solution to the differential equation

$$
x^{4} \frac{d y}{d x}+x^{2} y^{2}=6 x^{4}-14 x^{2}+4
$$

## Solution:

$$
y=a x+\frac{b}{x} \Rightarrow \frac{d y}{d x}=a-\frac{b}{x^{2}} \text { and } y^{2}=a^{2} x^{2}+2 a b+\frac{b^{2}}{x^{2}}
$$

So

$$
\begin{gathered}
x^{4} \frac{d y}{d x}+x^{2} y^{2}=6 x^{4}-14 x^{2}+4 \\
\Leftrightarrow x^{4}\left(a-\frac{b}{x^{2}}\right)+x^{2}\left(a^{2} x^{2}+2 a b+\frac{b^{2}}{x^{2}}\right)=6 x^{4}-14 x^{2}+4 \\
\Leftrightarrow\left(a^{2}+a\right) x^{4}+(2 a b-b) x^{2}+b^{2}=6 x^{4}-14 x^{2}+4 \\
\Leftrightarrow a^{2}+a=6,2 a b-b=14 \text { and } b^{2}=4 \\
\Leftrightarrow a=-3 \text { or } 2, b=2 \text { or }-2 \text { and } 2 a b-b=14 \\
\Leftrightarrow a=-3 \text { and } b=2 .
\end{gathered}
$$

So the only possibility is that

$$
y=-3 x+\frac{2}{x}
$$

2. [10 marks] Newton's Law of Cooling, or Heating, states that

$$
\frac{d T}{d t}=-k(T-A)
$$

for some positive constant $k$, where $T$ is the temperature of an object at time $t$ and $A$ is the temperature, presumed to be constant, of the surrounding medium.

Suppose that at 10 AM a cup of coffee with temperature $90^{\circ} \mathrm{C}$ is placed on a table in a kitchen with constant air temperature $20^{\circ} \mathrm{C}$. If the temperature of the coffee is $60^{\circ} \mathrm{C}$ five minutes later, when will the temperature of the coffee be $25^{\circ} \mathrm{C}$ ?

Solution: Let $t=0$ be 10 AM and take $A=20$. Separate variables.

$$
\begin{aligned}
\int \frac{d T}{T-20} & =-\int k d t \\
\Leftrightarrow \ln |T-20| & =-k t+C
\end{aligned}
$$

To find $C$ let $t=0$ and $T=90$ :

$$
\ln 70=0+C \Leftrightarrow C=\ln 70 .
$$

To find $k$ let $t=5, T=60, C=\ln 70$ :

$$
\begin{aligned}
\ln 40 & =-5 k+\ln 70 \\
\Leftrightarrow 5 k & =\ln 70-\ln 40 \\
\Leftrightarrow k & =\frac{1}{5} \ln \frac{7}{4} \simeq 0.111923157 \ldots
\end{aligned}
$$

Finally, let $T=25$ and solve for $t$ :

$$
\begin{aligned}
\ln 5=-\frac{t}{5} \ln \frac{7}{4}+\ln 70 & \Rightarrow \frac{t}{5}=\ln 70-\ln 5 \\
& \Rightarrow t=\frac{5 \ln 14}{\ln 1.75} \simeq 23.579189 \ldots
\end{aligned}
$$

So the temperature of the coffee will be $25^{\circ} \mathrm{C}$ at 10:23:35 AM
3. [10 marks] Solve for $x$ as a function of $t$ if $x^{\prime \prime}+8 x^{\prime}+25 x=0$ and $x(0)=2, x^{\prime}(0)=-4$.

Solution: the auxiliary quadratic is $r^{2}+8 r+25$. Solve:

$$
r^{2}+8 r+25=0 \Leftrightarrow r=\frac{-8 \pm \sqrt{64-100}}{2}=-4 \pm 3 i .
$$

Thus

$$
x=C_{1} e^{-4 t} \cos (3 t)+C_{2} e^{-4 t} \sin (3 t) .
$$

To find $C_{1}$ use the initial condition $x=2$ when $t=0$ :

$$
2=C_{1} e^{0} \cos 0+C_{2} e^{0} \sin 0 \Leftrightarrow C_{1}=2 .
$$

To find $C_{2}$ you need to find $x^{\prime}$ :

$$
x^{\prime}=C_{1}\left(-4 e^{-4 t} \cos (3 t)-3 e^{-2 t} \sin (3 t)\right)+C_{2}\left(-4 e^{-4 t} \sin (3 t)+3 e^{-4 t} \cos (3 t)\right) .
$$

Now substitute $t=0, x^{\prime}=-4, C_{1}=2$ :

$$
-4=2(-4-0)+C_{2}(0+3) \Leftrightarrow C_{2}=\frac{4}{3}
$$

Thus

$$
x=2 e^{-4 t} \cos (3 t)+\frac{4}{3} e^{-4 t} \sin (3 t)
$$

4. [10 marks] Decide if the following infinite series converge or diverge; justify your choice.
(a) [3 marks] $\sum_{k=0}^{\infty} \frac{4 k^{1 / 2}+3 k}{2 k^{5 / 2}-k^{2}+k+10}$

Calculation: let

$$
a_{k}=\frac{4 k^{1 / 2}+3 k}{2 k^{5 / 2}-k^{2}+k+10} b_{k}=\frac{1}{k^{3 / 2}} .
$$

Then

$$
L=\lim _{k \rightarrow \infty} \frac{a_{k}}{b_{k}}=\lim _{k \rightarrow \infty} \frac{4 k^{2}+3 k^{5 / 2}}{2 k^{5 / 2}-k^{2}+k+10}=\frac{3}{2} .
$$

Since $\sum_{k=0}^{\infty} b_{k}$ converges, because it is a $p$-series with $p=3 / 2>1$, and $0<L<\infty$,

$$
\sum_{k=0}^{\infty} a_{k}
$$

converges, by the Limit Comparison Test.
(b) $[3$ marks $] \sum_{k=2}^{\infty} \frac{1}{k \ln k}$
by the integral test

## Calculation:

$$
\int_{2}^{\infty} \frac{d x}{x \ln x}=\int_{\ln 2}^{\infty} \frac{d u}{u}=\lim _{b \rightarrow \infty}[\ln u]_{\ln 2}^{b}=\infty
$$

(c) $[4$ marks $] \sum_{k=1}^{\infty} \frac{k^{k}}{k!}$

## Calculation:

$$
\lim _{k \rightarrow \infty} \frac{a_{k+1}}{a_{k}}=\lim _{k \rightarrow \infty} \frac{(k+1)^{k+1}}{k^{k}} \frac{k!}{(k+1)!}=\lim _{k \rightarrow \infty}\left(\frac{k+1}{k}\right)^{k}=e>1 .
$$

5. [10 marks] If $x$ is the mass of salt dissolved in a saline solution of volume $V$, at time $t$, in a large mixing tank, then

$$
\frac{d x}{d t}+\frac{r_{o}}{V} x=r_{i} c_{i}
$$

where $c_{i}$ is the concentration of salt in a solution entering the mixing tank at rate $r_{i}$, and $r_{o}$ is the rate at which the well-mixed solution is leaving the tank.
A 100-liter tank initially contains 50 liters of brine (i.e. saline solution) containing 10 kg of salt. Brine containing 1 kg of salt per liter enters the tank at the rate of 5 liters per sec , and the well-mixed brine in the tank flows out at the rate of 3 liters per sec. How many kilograms of salt will the tank contain when it is full?

Solution: $V=50+\left(r_{i}-r_{o}\right) t=50+2 t$. The integrating factor of the differential equation

$$
\frac{d x}{d t}+\frac{3 x}{50+2 t}=5
$$

is

$$
\mu=e^{\int \frac{3}{50+2 t} d t}=e^{3 \ln (25+t) / 2}=(25+t)^{3 / 2}
$$

and so

$$
x=\frac{\int 5 \mu d t}{\mu}=\frac{2(25+t)^{5 / 2}+C}{(25+t)^{3 / 2}}=50+2 t+\frac{C}{(25+t)^{3 / 2}} .
$$

Use the initial condition $t=0, x=10$ to find $C$ :

$$
10=50+\frac{C}{25^{3 / 2}} \Leftrightarrow C=-40 \cdot 125=-5000 .
$$

Thus

$$
x=(50+2 t)-\frac{5000}{(25+t)^{3 / 2}} .
$$

The tank is full when $V=100 \Leftrightarrow 50+2 t=100 \Leftrightarrow t=25$. At $t=25$,

$$
x=(50+50)-\frac{5000}{(50)^{3 / 2}}=100-\frac{20}{\sqrt{2}} \simeq 85.9 .
$$

So when the tank is full of brine it contains about 85.9 kilograms of salt.
6. [10 marks; 5 for each part.] Find the sum of the following infinite series:
(a) $\sum_{k=0}^{\infty}\left(\frac{1}{3^{k}}-\frac{2}{5^{k+2}}\right)$

Solution: use the sum of an infinite geometric series, $\sum_{k=0}^{\infty} r^{k}=\frac{1}{1-r}$, if $|r|<1$.

$$
\begin{aligned}
\sum_{k=0}^{\infty}\left(\frac{1}{3^{k}}-\frac{2}{5^{k+2}}\right) & =\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{k}-\frac{2}{25} \sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^{k} \\
& =\frac{1}{1-1 / 3}-\frac{2}{25} \frac{1}{1-1 / 5} \\
& =\frac{3}{2}-\frac{2}{25} \cdot \frac{5}{4} \\
& =\frac{3}{2}-\frac{1}{10}=\frac{7}{5}=1.4
\end{aligned}
$$

(b) $\sum_{k=3}^{\infty} \frac{4}{k^{2}-4}$

Solution: use partial fractions.

$$
\frac{4}{k^{2}-4}=\frac{1}{k-2}-\frac{1}{k+2}
$$

So $\sum_{k=3}^{\infty} \frac{4}{k^{2}-4}$ is a telescoping sum:

$$
\begin{aligned}
\sum_{k=3}^{\infty} \frac{4}{k^{2}-4} & =\sum_{k=3}^{\infty}\left(\frac{1}{k-2}-\frac{1}{k+2}\right) \\
& =\frac{1}{1}-\frac{1}{5}+\frac{1}{2}-\frac{1}{6}+\frac{1}{3}-\frac{1}{7}+\frac{1}{4}-\frac{1}{8}+\frac{1}{5}-\frac{1}{9}+\frac{1}{6}-\frac{1}{10} \cdots \\
& =1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \\
& =\frac{25}{12}
\end{aligned}
$$

