University of Toronto
Solutions to **MAT187H1F TERM TEST**
of **THURSDAY, JUNE 3, 2010**
Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all seven questions. Present your solutions in the booklets provided. The value for each question is indicated in parantheses beside the question number.
TOTAL MARKS: 60

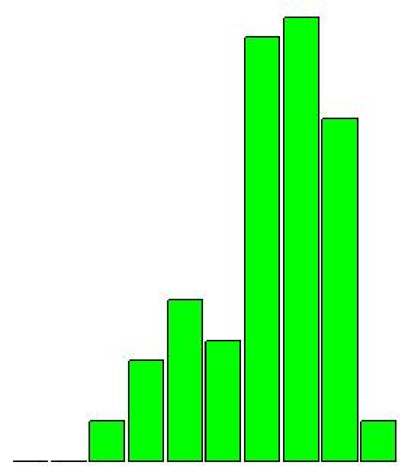
General Comments:

1. The questions on this test are considered routine. The only possible complications are in Question 2, solving for y; and in Question 7(b), which is a telescoping sum in which four terms do not cancel.

2. The results on this test were better than the results on the first test.

Breakdown of Results: 83 registered students wrote this test. The marks ranged from 20.8% to 95.8%, and the average was 66.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.4%
A	22.9%	80-89%	20.5%
B	26.5%	70-79%	26.5%
C	25.3%	60-69%	25.3%
D	7.2%	50-59%	7.2%
F	18.1%	40-49%	9.7%
		30-39%	6.0%
		20-29%	2.4%
		10-19%	0.0%
		0-9%	0.0%



1. [6 marks] Suppose that the population of a bacteria colony is growing exponentially so that its population doubles every 4 hours. How long does it take for the bacteria colony to triple its population?

Solution: for exponential growth the doubling time is $t = \dfrac{\ln 2}{k}$. So

$$4 = \frac{\ln 2}{k} \Leftrightarrow k = \frac{\ln 2}{4} \simeq 0.1732867952.$$

Then

$$3x_0 = x_0 e^{kt} \Leftrightarrow e^{kt} = 3 \Leftrightarrow t = \frac{\ln 3}{k} \simeq 6.339850003;$$

so the tripling time is about 6.34 hours.

2. [10 marks] Solve the initial value problem

$$\text{DE: } (1+2y^2)\, y' = 2\, x\, y^2 \quad \text{IC: } y(0)=1.$$

for y as an explicit function of x.

Solution: separate variables.

$$\begin{aligned}
(1+2y^2)\frac{dy}{dx} &= 2xy^2 \\
\Leftrightarrow \int \left(\frac{1}{y^2}+2\right)\, dy &= \int 2x\, dx \\
\Leftrightarrow -\frac{1}{y}+2y &= x^2+C
\end{aligned}$$

To find C let $x=0$ and $y=1$:

$$-1+2 = 0 + C \Leftrightarrow C = 1.$$

Thus

$$-\frac{1}{y}+2y = x^2+1.$$

Isolating y is the hardest part:

$$\begin{aligned}
-\frac{1}{y}+2y = x^2+1 \quad &\Rightarrow \quad -1+2y^2 = (x^2+1)y \\
&\Rightarrow \quad 2y^2-(x^2+1)y-1=0 \\
&\Rightarrow \quad y = \frac{x^2+1 \pm \sqrt{(x^2+1)^2+8}}{4} \\
(\text{using IC: } y(0)=1) \quad &\Rightarrow \quad y = \frac{x^2+1 + \sqrt{(x^2+1)^2+8}}{4}
\end{aligned}$$

3. [8 marks] Find the general solution to the differential equation

$$\frac{dy}{dx} - 2xy = 2x.$$

What is the particular solution that passes through the point $(0, 3)$?

Solution: use the method of the integrating factor.

$$\mu = e^{\int (-2x)\,dx} = e^{-x^2}.$$

The general solution is

$$\begin{aligned} y &= \frac{\int \mu\, 2x\, dx}{\mu} \\ &= e^{x^2} \int 2x\, e^{-x^2}\, dx \\ &= e^{x^2} \left(-e^{-x^2} + C \right) \\ &= -1 + Ce^{x^2} \end{aligned}$$

Particular Solution: if $x = 0$ and $y = 3$, then

$$3 = -1 + C \Rightarrow C = 4.$$

So the particular solution passing through the point $(0, 3)$ is

$$y = -1 + 4e^{x^2}.$$

4. [9 marks] Solve for x as a function of t if $x'' + 4x' + 5x = 0$ and $x(0) = 2, x'(0) = -4$.

Solution: the auxiliary quadratic is $r^2 + 4r + 5$. Solve:

$$r^2 + 4r + 5 = 0 \Leftrightarrow r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.$$

Thus

$$x = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t.$$

To find C_1 use the initial condition $x = 2$ when $t = 0$:

$$2 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 2.$$

To find C_2 you need to find x' :

$$x' = C_1(-2e^{-2t} \cos t - e^{-2t} \sin t) + C_2(-2e^{-2t} \sin t + e^{-2t} \cos t).$$

Now substitute $t = 0, x' = -4, C_1 = 2$:

$$-4 = 2(-2 - 0) + C_2(0 + 1) \Leftrightarrow C_2 = 0.$$

Thus

$$x = 2e^{-2t} \cos t.$$

5. [9 marks; 3 for each part.] Decide if the following infinite series converge or diverge; justify your choice.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{5}{3^k+1}$ $\otimes$ Converges $\bigcirc$ Diverges

by the limit comparison test

Calculation: use the fact that the infinite geometric series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k}$ converges.

$$a_k = \frac{5}{3^k+1},\ b_k = \frac{1}{3^k} \text{ and } \lim_{k\to\infty} \frac{a_k}{b_k} = 5 \lim_{k\to\infty} \frac{3^k}{3^k+1} = 5 \lim_{k\to\infty} \frac{1}{1+3^{-k}} = 5.$$

Or use comparison test:

$$a_k < \frac{5}{3^k} \text{ and } \sum_{k=0}^{\infty} \frac{5}{3^k} = 5 \sum_{k=0}^{\infty} \frac{1}{3^k}$$

converges.

(b) $\displaystyle\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$ $\bigcirc$ Converges $\otimes$ Diverges

by the k-th term test

Calculation:

$$\lim_{k\to\infty} k^2 \sin^2\left(\frac{1}{k}\right) = \lim_{k\to\infty} \frac{\sin^2\left(\frac{1}{k}\right)}{\frac{1}{k^2}} = \lim_{h\to 0} \frac{\sin^2 h}{h^2} = \left(\lim_{h\to 0} \frac{\sin h}{h}\right)^2 = 1 \neq 0.$$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{e^k}$ $\otimes$ Converges $\bigcirc$ Diverges

by the ratio test

Calculation:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k} \frac{e^k}{e^{k+1}} = \frac{1}{e} \lim_{k\to\infty} \frac{k}{k+1} = \frac{1}{e} < 1.$$

6. [10 marks] If x is the mass of salt dissolved in a saline solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A 100-liter tank initially contains 70 liters of brine (i.e. saline solution) containing 10 kg of salt. Brine containing 1 kg of salt per liter enters the tank at the rate of 5 liters per sec, and the well-mixed brine in the tank flows out at the rate of 2 liters per sec. How many kilograms of salt will the tank contain when it is full?

Solution: $V = 70 + (r_i - r_o)t = 70 + 3t$. The integrating factor of the differential equation

$$\frac{dx}{dt} + \frac{2}{70+3t}x = 5$$

is

$$\mu = e^{\int \frac{2}{70+3t}\,dt} = e^{2\ln(70+3t)/3} = (70+3t)^{2/3}$$

and so

$$x = \frac{\int 5\,\mu\,dt}{\mu} = \frac{(70+3t)^{5/3}+C}{(70+3t)^{2/3}} = (70+3t) + \frac{C}{(70+3t)^{2/3}}.$$

Use the initial condition $t = 0, x = 10$ to find C :

$$10 = 70 + \frac{C}{70^{2/3}} \Leftrightarrow C = -60 \cdot 70^{2/3} \simeq -1019.1.$$

Thus

$$x = (70+3t) - \frac{60 \cdot 70^{2/3}}{(70+3t)^{2/3}}.$$

The tank is full when $V = 100 \Leftrightarrow 70 + 3t = 100 \Leftrightarrow t = 10$. At $t = 10$,

$$x = (70+30) - \frac{60 \cdot 70^{2/3}}{(70+30)^{2/3}} = 100 - 60\left(\frac{7}{10}\right)^{2/3} \simeq 52.7.$$

So when the tank is full of brine it contains about 52.7 kilograms of salt.

7. [8 marks; 4 for each part.] Find the sum of the following infinite series:

(a) $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{3^k}+\frac{2}{5^{k+1}}\right)$

Solution: use the sum of an infinite geometric series, $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, if $|r|<1$.

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{1}{3^k}+\frac{2}{5^{k+1}}\right) &= \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k+\frac{2}{5}\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k \\ &= \frac{1}{1-1/3}+\frac{2}{5}\frac{1}{1-1/5} \\ &= \frac{3}{2}+\frac{2}{5}\cdot\frac{5}{4} \\ &= \frac{3}{2}+\frac{2}{4}=2\end{aligned}$$

(b) $\displaystyle\sum_{k=5}^{\infty}\frac{4}{k^2-4k}$

Solution: use partial fractions.

$$\frac{4}{k^2-4k}=\frac{1}{k-4}-\frac{1}{k}$$

So $\displaystyle\sum_{k=5}^{\infty}\frac{4}{k^2-4k}$ is a telescoping sum:

$$\begin{aligned}\sum_{k=5}^{\infty}\frac{4}{k^2-4k} &= \sum_{k=5}^{\infty}\left(\frac{1}{k-4}-\frac{1}{k}\right) \\ &= \frac{1}{1}-\frac{1}{5}+\frac{1}{2}-\frac{1}{6}+\frac{1}{3}-\frac{1}{7}+\frac{1}{4}-\frac{1}{8}+\frac{1}{5}-\frac{1}{9}+\frac{1}{6}-\frac{1}{10}\cdots \\ &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \\ &= \frac{25}{12}\end{aligned}$$