University of Toronto
Solutions to **MAT187H1F TERM TEST**
of **WEDNESDAY, JUNE 10, 2009**
Duration: 90 minutes

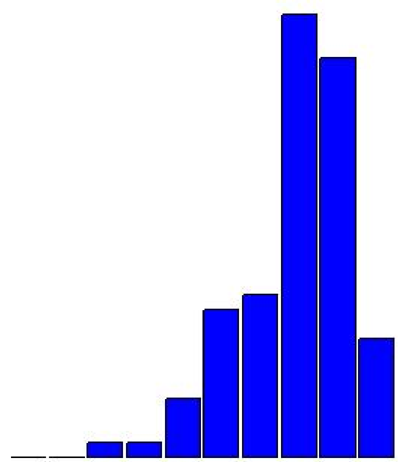
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all seven questions. Present your solutions in the booklets provided. The value for each question is indicated in parantheses beside the question number. **TOTAL MARKS: 60**

General Comments: The results on this test were very good, and very few students failed.

Breakdown of Results: 92 registered students wrote this test. The marks ranged from 28.3% to 93.3%, and the average was 73.15%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.7%
A	38.0%	80-89%	29.3%
B	32,8%	70-79%	32.6%
C	12.0%	60-69%	12.0%
D	10.9%	50-59%	10.9%
F	6.5%	40-49%	4.3%
		30-39%	1.1%
		20-29%	1.1 %
		10-19%	0.0%
		0-9%	0.0%



1. [8 marks] Consider the curve with parametric equations $x = e^{t^2}; y = t^2 - t^3$. Find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(x, y) = (e, 0)$, for which $t = 1$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 3t^2}{2te^{t^2}} = e^{-t^2} - \frac{3}{2}te^{-t^2}.$$

So at $t = 1$,

$$\frac{dy}{dx} = e^{-1} - \frac{3}{2}e^{-1} = -\frac{1}{2e}.$$

Also,

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(e^{-t^2} - \frac{3}{2}te^{-t^2}\right)}{2te^{t^2}} = \frac{-2te^{-t^2} - \frac{3}{2}e^{-t^2} + 3t^2e^{-t^2}}{2te^{t^2}}.$$

So at $t = 1$,

$$\frac{d^2y}{dx^2} = \frac{-2e^{-1} - 3e^{-1}/2 + 3e^{-1}}{2e} = -\frac{1}{4e^2}.$$

2. [8 marks] Solve for y as a function of x if

$$3y'' + 6y' + 6\,y = 0;\; y(0) = 2,\; y'(0) = -4.$$

Solution: the auxiliary quadratic is $3r^2 + 6r + 6$. Solve:

$$3r^2 + 6r + 6 = 0 \Leftrightarrow r = \frac{-6 \pm \sqrt{36 - 72}}{6} = -1 \pm i.$$

Thus

$$y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x.$$

To find C_1 use the initial condition $y = 2$ when $x = 0$:

$$2 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 2.$$

To find C_2 you need to find y' :

$$y' = C_1(-e^{-x} \cos x + e^{-x} \sin x) + C_2(-e^{-x} \sin x + e^{-x} \cos x)$$

Now substitute $x = 0, y' = -4, C_1 = 2$:

$$-4 = -(2) + C_2 \Leftrightarrow C_2 = -4 + 2 \Leftrightarrow C_2 = -2.$$

Thus

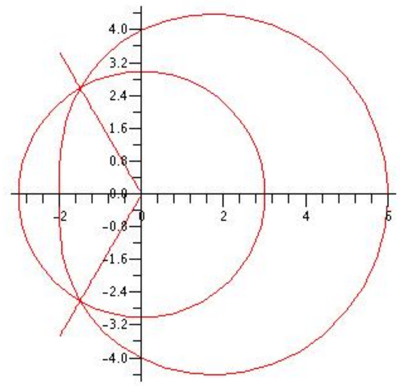
$$y = 2e^{-x} \cos x - 2e^{-x} \sin x.$$

3. [8 marks] Find the area of the region inside the curve with polar equation $r = 4 + 2\cos\theta$ but outside the circle with polar curve $r = 3$.

Solution:

Find the intersection points:

$$4 + 2\cos\theta = 3 \;\Rightarrow\; \cos\theta = -\frac{1}{2}$$
$$\Rightarrow\; \theta = \pm\frac{2\pi}{3}$$



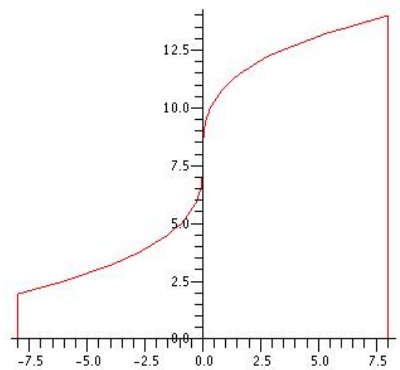
Then

$$\begin{aligned}
A &= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} \left((4+2\cos\theta)^2 - 3^2\right)\, d\theta \\
&= 2\left(\frac{1}{2}\right)\int_{0}^{2\pi/3} \left((4+2\cos\theta)^2 - 3^2\right)\, d\theta, \text{ by symmetry} \\
&= \int_{0}^{2\pi/3} \left(7 + 16\cos\theta + 4\cos^2\theta\right)\, d\theta \\
&= \int_{0}^{2\pi/3} (7 + 16\cos\theta + 2(1+\cos 2\theta))\, d\theta, \text{ using double angle formula} \\
&= \int_{0}^{2\pi/3} (9 + 16\cos\theta + 2\cos 2\theta)\, d\theta \\
&= [9\,\theta + 16\sin\theta + \sin 2\theta]_0^{2\pi/3} \\
&= 6\pi + 16\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} \\
&= 6\pi + \frac{15\sqrt{3}}{2}
\end{aligned}$$

4. [8 marks] Consider the curve with parametric equations $x = t^3$ and $y = 3t + 8$ for $t \in \mathbb{R}$. Find the area of the region under the curve and above the x-axis, for $-8 \leq x \leq 8$.

Solution:

$$x = \pm 8 \Rightarrow t^3 = \pm 8 \Rightarrow t = \pm 2$$



Then:

$$\begin{aligned}
A &= \int_{-8}^{8} y\,dx \\
&= \int_{-2}^{2} (3t+8)\,3t^2\,dt \\
&= \int_{-2}^{2} \left(9t^3 + 24t^2\right)\,dt \\
&= \left[\frac{9}{4}t^4 + 8t^3\right]_{-2}^{2} \\
&= 36 + 64 - 36 - (-64) \\
&= 128
\end{aligned}$$

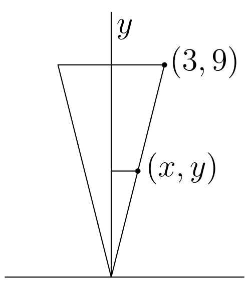
5. [10 marks] Torricelli's Law states that

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

where y is the depth of a fluid in a tank at time t, $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and g is the acceleration due to gravity.

An inverted conical water tank with a radius at its top of 3 m and a height of 9 m above its vertex, is initially full. At 12 noon a plug at the bottom of the tank is removed, and 10 min later the depth of the water in the tank is 4 m. When will the tank be completely empty?

Solution:



By similar triangles, $\frac{x}{y} = \frac{3}{9} \Rightarrow x = \frac{y}{3}$, so

$$\begin{aligned} A(y) &= \pi x^2 \\ &= \pi\left(\frac{y}{3}\right)^2 \\ &= \frac{\pi}{9}y^2 \end{aligned}$$

Solve the DE by separating variables:

$$\begin{aligned} A(y)\frac{dy}{dt} = -a\sqrt{2gy} \quad &\Leftrightarrow \quad \frac{\pi}{9}\int \frac{y^2}{\sqrt{y}}\,dy = -\frac{\pi}{9}\int K\,dt, \text{ for } K = \frac{9a\sqrt{2g}}{\pi} \\ &\Leftrightarrow \quad \int y^{3/2}\,dy = -\int K\,dt \\ &\Leftrightarrow \quad \frac{2}{5}y^{5/2} = -Kt + C, \text{ for some } C \end{aligned}$$

Let t be measured in minutes; let $t = 0$ be noon. When $t = 0, y = 9$, so

$$\frac{2}{5}9^{5/2} = C \Leftrightarrow C = \frac{486}{5} = 97.2$$

When $t = 10, y = 4$, so

$$\frac{2}{5}4^{5/2} = -10K + \frac{486}{5} \Leftrightarrow 10K = \frac{486}{5} - \frac{64}{5} \Leftrightarrow K = \frac{211}{25} = 8.44$$

The tank is empty when $y = 0$:

$$0 = -Kt + C \Leftrightarrow t = \frac{C}{K} = \frac{97.2}{8.44} = \frac{2430}{211} \simeq 11.5.$$

So the tank will be empty at about 12:11:30 PM; that is, a minute and a half after 12.10 PM.

6. [10 marks] Find the particular solution to the initial value problem:

$$\text{DE: } \frac{dy}{dx} + \frac{3\,y}{x} = \frac{\cos x}{x^2}; \quad \text{IC: } y = 0 \text{ if } x = -\pi.$$

Solution: The integrating factor is

$$\rho = e^{\int \frac{3}{x}\,dx} = e^{3\ln|x|} = |x|^3;$$

take either $\pm x^3$. Then the general solution is

$$\begin{aligned} y &= \frac{\int \rho \frac{\cos x}{x^2}\,dx}{\rho} \\ &= \frac{\int x\cos x\,dx}{x^3} \\ &= \frac{x\sin x + \cos x + C}{x^3}, \text{ by parts,} \end{aligned}$$

To find C, let $x = -\pi$ and $y = 0$:

$$0 = \frac{0 - 1 + C}{-\pi^3} \Leftrightarrow C = 1.$$

Thus the particular solution is

$$y = \frac{x\sin x + \cos x + 1}{x^3} \text{ or } \frac{\sin x}{x^2} + \frac{\cos x}{x^3} + \frac{1}{x^3}.$$

7. [8 marks] Find the length of the curve with parametric equations

$$x = \ln t;\ y = 2t;\ z = t^2$$

for $1 \leq t \leq 4$.

Solution:

$$\begin{aligned}
L &= \int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_1^4 \sqrt{\left(\frac{1}{t}\right)^2 + (2)^2 + (2t)^2}\, dt \\
&= \int_1^4 \sqrt{\frac{1}{t^2} + 4 + 4t^2}\, dt \\
&= \int_1^4 \sqrt{\left(\frac{1}{t} + 2t\right)^2}\, dt \\
&= \int_1^4 \left(\frac{1}{t} + 2t\right) dt, \text{ since } t > 0 \\
&= \left[\ln t + t^2\right]_1^4 \\
&= \ln 4 + 15
\end{aligned}$$