University of Toronto
Solutions to **MAT 187H1F TERM TEST**
of **FRIDAY, JUNE 6, 2008**
Duration: 90 minutes

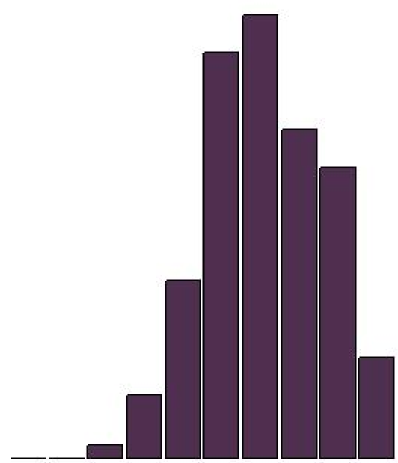
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Make sure this test contains 8 pages. Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parantheses beside the question number. Do not tear out any pages.

General Comments: the results on this test were not as good as on the first test. Nevertheless, questions 1, 2, 3 and 6 are considered very straightforward, routine questions – they should all have been aced! Questions 5 and 7 are applications, very similar to examples I did in class; however many students had difficulty with these two questions. Finally, question 4 caused difficulty as well, probably because it required the formula for surface area – but it was based on an assigned homework problem. Only one number was changed, to make the integral work out even easier.

Breakdown of Results: 144 registered students wrote this test. The marks ranged from 23% to 97%, and the average was 65.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.6%
A	21.5%	80-89%	15.9%
B	18.1%	70-79%	18.1%
C	24.3%	60-69%	24.3%
D	22.2%	50-59%	22.2%
F	13.9%	40-49%	9.7%
		30-39%	3.5%
		20-29%	0.7%
		10-19%	0.0%
		0-9%	0.0 %



1. [8 marks] Consider the curve with parametric equations

$$x = 1 - 3t^2; y = 3t - t^3.$$

Find both

$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}$$

in terms of the parameter t.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3-3t^2}{-6t} = -\frac{1}{2t} + \frac{t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\frac{1}{2t} + \frac{t}{2}\right)}{-6t} = \frac{\frac{1}{2t^2} + \frac{1}{2}}{-6t} = -\frac{1+t^2}{12t^3}$$

2. [8 marks] Solve for y as a function of x if

$$25y'' + 14y' + 25\,y = 0;\; y(0) = -4,\; y'(0) = 4.$$

Solution: the auxiliary quadratic is $25r^2 + 14r + 25$. Solve:

$$25r^2 + 14r + 25 = 0 \Leftrightarrow r = \frac{-14 \pm \sqrt{196 - 2500}}{2} = \frac{-7 \pm 24i}{25} = -0.28 \pm 0.96i.$$

Thus

$$y = C_1 e^{-.28x}\cos(.96x) + C_2 e^{-.28x}\sin(.96x).$$

To find C_1 use the initial condition $y = -4$ when $x = 0$:

$$-4 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = -4.$$

To find C_2 you need to find y' :

$$y' = C_1(-.28e^{-.28x}\cos .96x - .96e^{-2x}\sin .96x) + C_2(-.28e^{-.28x}\sin .96x + .96e^{-.28x}\cos .96x)$$

Now substitute $x = 0, y' = 4, C_1 = -4$:

$$4 = -.28C_1 + .96C_2 \Leftrightarrow .96C_2 = 4 - 1.12 \Leftrightarrow C_2 = 3.$$

Thus

$$y = -4e^{-.28x}\cos(.96x) + 3e^{-.28x}\sin(.96x).$$

3. [8 marks] Plot the polar curve with polar equation $r = \cos(2\theta)$ for

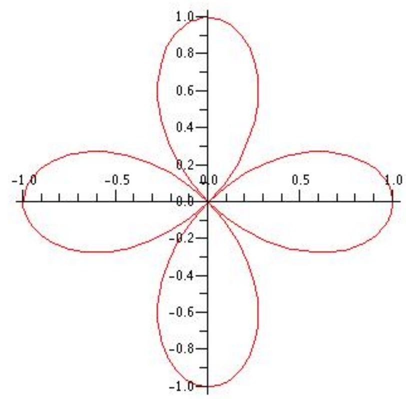
$$0 \le \theta \le 2\pi$$

and find the total area within the curve.

Solution:

This is a four leaved rose.

$$\begin{aligned} r = 0 \quad &\Rightarrow \quad \cos 2\theta = 0 \\ &\Rightarrow \quad 2\theta = \pm\frac{\pi}{2} \text{ or } \pm\frac{3\pi}{2} \\ &\Rightarrow \quad \theta = \pm\frac{\pi}{4} \text{ or } \pm\frac{3\pi}{4} \end{aligned}$$



To find total area within the curve, you can take four times the area of one loop, or you can simply integrate right around, from $\theta = 0$ to $\theta = 2\pi$.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} r^2\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \cos^2 2\theta\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \frac{1+\cos(4\theta)}{2}\, d\theta \\ &= \frac{1}{4}\left[\theta + \frac{\sin(4\theta)}{4}\right]_0^{2\pi} \\ &= \frac{1}{4}(2\pi + 0 - 0 - 0) \\ &= \frac{\pi}{2} \end{aligned}$$

4. [10 marks] Consider the curve with parametric equations

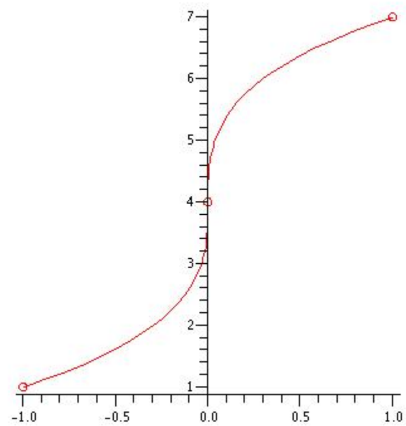
$$x = t^3, y = 3t + 4, \text{ for } -1 \leq t \leq 1.$$

Find the area of the surface of revolution generated by revolving the above curve around the y-axis.

Solution:

$$\begin{aligned} t = -1 &\Rightarrow (x, y) = (-1, 1) \\ t = 0 &\Rightarrow (x, y) = (0, 4) \\ t = 1 &\Rightarrow (x, y) = (1, 7) \end{aligned}$$

Note: $x < 0 \Leftrightarrow t < 0$.



To find the total surface area of the surface of revolution generated by revolving the above curve around the y-axis, you can double the area of the top half of the surface, for $0 \leq t \leq 1 \Leftrightarrow 0 \leq x \leq 1$, by symmetry.

$$\begin{aligned} SA &= 2\int_0^1 2\pi x \, ds \\ &= \pi \int_0^1 4t^3 \sqrt{(3t^2)^2 + 3^2} \, dt \\ &= 3\pi \int_0^1 4t^3 \sqrt{t^4 + 1} \, dt \\ &= 3\pi \int_1^2 \sqrt{u} \, du, \text{ with } u = t^4 + 1 \\ &= 3\pi \left[\frac{2}{3} u^{3/2}\right]_1^2 \\ &= 2\pi(2\sqrt{2} - 1) \end{aligned}$$

Note: if you don't double the surface area of the top half, then

$$SA = \int_{-1}^1 2\pi |x| \, ds = \int_{-1}^1 2\pi |t^3| \sqrt{9t^4 + 9} \, dt,$$

for which you must take two cases: $-1 \leq t \leq 0$, and $0 \leq t \leq 1$, and use the fact that $|t^3| = -t^3$ in the former case. Otherwise, your integral will be zero.

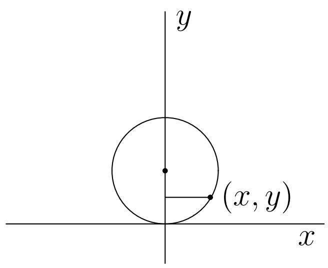
5. [10 marks] Torricelli's Law states that

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

where y is the depth of a fluid in a tank at time t, $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and g is the acceleration due to gravity.

A spherical water tank of radius 1 m is initially full. At 12 noon a plug at the bottom of the tank is removed, and 10 min later the tank is half empty. When will the tank be completely empty?

Solution:



$x^2 + (y-1)^2 = 1$

$$\begin{aligned} A(y) &= \pi x^2 \\ &= \pi\left(1-(y-1)^2\right) \\ &= \pi(1-y^2+2y-1) \\ &= \pi(2y-y^2) \end{aligned}$$

Solve the DE by separating variables:

$$\begin{aligned} A(y)\frac{dy}{dt} = -a\sqrt{2gy} \quad &\Leftrightarrow \quad \pi\int \frac{2y-y^2}{\sqrt{y}}\,dy = -\pi\int K\,dt, \text{ for } K = \frac{a\sqrt{2g}}{\pi} \\ &\Leftrightarrow \quad \int \left(2\sqrt{y} - y^{3/2}\right)\,dy = -\int K\,dt \\ &\Leftrightarrow \quad \frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -Kt + C, \text{ for some } C \end{aligned}$$

Let t be measured in minutes; let $t=0$ be noon. When $t=0, y=2$, so

$$\frac{4}{3}2^{3/2} - \frac{2}{5}2^{5/2} = C \Leftrightarrow C = \frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2} = \frac{16}{15}\sqrt{2}.$$

When $t = 10, y = 1$, so

$$\frac{4}{3} - \frac{2}{5} = -10K + \frac{16}{15}\sqrt{2} \Leftrightarrow 10K = \frac{16}{15}\sqrt{2} - \frac{14}{15} \Leftrightarrow K = \frac{8\sqrt{2}-7}{75}$$

The tank is empty when $y = 0:$

$$0 = -Kt + C \Leftrightarrow t = \frac{C}{K} = \frac{80\sqrt{2}}{8\sqrt{2}-7} \simeq 26.2.$$

So the tank will be empty at about 12:26 PM.

6. [8 marks] Find the general solution to the differential equation

$$\frac{dy}{dx} + \frac{y}{1+x} = \frac{\cos x}{1+x}.$$

Solution: The integrating factor is

$$\rho = e^{\int \frac{1}{1+x}\,dx} = e^{\ln|1+x|} = |1+x|;$$

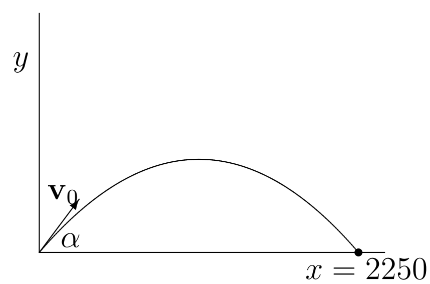
take either $\pm(1+x)$. Then the general solution is

$$\begin{aligned} y &= \frac{\int \rho \frac{\cos x}{1+x}\,dx}{\rho} \\ &= \frac{\int \cos x\,dx}{1+x} \\ &= \frac{\sin x + C}{1+x} \\ &= \frac{\sin x}{1+x} + \frac{C}{1+x} \end{aligned}$$

7. [8 marks] A cannon ball is fired from a cannon with initial speed 150 m/sec. At what angles to the horizontal can the cannon be aimed if the cannon ball is to hit a target 2,250 metres down range? (Use $g = 9.8\text{m/sec}^2$ and ignore air resistance.)

Solution: Use $x = (v_0 \cos\alpha)t + x_0; y = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t + y_0$ with

$$v_0 = 150; x_0 = 0 \text{ and } y_0 = 0.$$



$$\begin{aligned}
y = 0 &\Rightarrow t = 0 \text{ or } t = \frac{2v_0 \sin\alpha}{g} \\
x = 2250 &\Rightarrow 2250 = \frac{2v_0^2 \cos\alpha \sin\alpha}{g} \\
&\Rightarrow \sin(2\alpha) = \frac{2250g}{v_0^2} \\
&\Rightarrow \sin(2\alpha) = 0.98 \\
&\Rightarrow 2\alpha \simeq 78.52^\circ \text{ or } 101.48^\circ \\
&\Rightarrow \alpha \simeq 39.26^\circ \text{ or } 50.74^\circ
\end{aligned}$$