University of Toronto
Solutions to **MAT187H1S TERM TEST 1**
of **Thursday, January 30, 2014**
Duration: 100 minutes

Only aids permitted: Casio FX-991 or Sharp EL-520 calculator.
General Comments:

1. There are still many students who don't use = signs. How can anyone read your work?

2. Incredibly, there are still lots of students who think $1/(a+b) = 1/a + 1/b$. Shape up!

3. If you make a substitution, say $u = g(x)$, then after the substitution there should only be one variable in your integrand. It you write something like $\int f(x)\,du$ it is equal to $f(x)\,u + C$ because du indicates the variable of integration is u and so $f(x)$ is a constant with respect to u.

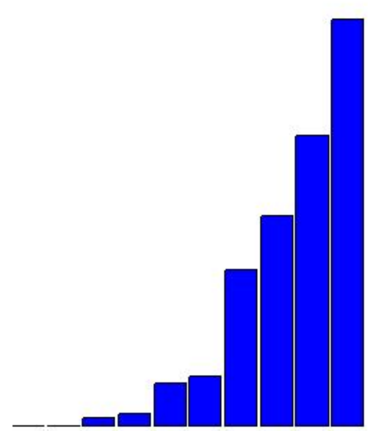
4. In Question 8 if you end up using the trig substitution $x^{2/3} = u = \sec\theta$ then it is only valid for $|x| > 1$, but the integral is defined for $x \neq 0, x \neq 1$, so this trig substitution will not give you the complete answer! To get the complete answer you would also have to consider $x^{2/3} = \sin\theta$.

5. If in Questions 1 or 3 you make a substitution in the definite integral, then you must change the limits of integration as you change the variable. Theorem 5.9.1 on page 391 of the textbook spells it out:
$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$
where the substitution is $u = g(x)$. Doing this can save you work. But in Question 3 it is actually easier to calculate the indefinite integral first.

6. The whole point of the Question 5 is how to handle the minus sign in $\sqrt{8+2x-x^2}$.
Since $8 + 2x - x^2 \geq 0 \Rightarrow 9 - (x-1)^2 \geq 0 \Rightarrow -3 \leq x - 1 \leq 3$, letting $x - 1 = 3\sec\theta$ or $3\tan\theta$ makes no sense at all.

Breakdown of Results: The results on this test were very good. 474 students wrote this test; only one student missed it! The marks ranged from 23% to 100%, and the average was 80%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right. There were 29 perfect papers.

Grade	%	Decade	%
		90-100%	34.6%
A	59.3%	80-89%	24.7%
B	17.9%	70-79%	17.9%
C	13.3%	60-69%	13.3 %
D	4.2%	50-59%	4.2%
F	5.3%	40-49%	3.6%
		30-39%	1.1%
		20-29%	0.6%
		10-19%	0.0 %
		0-9%	0.0%



Formulas you may find useful. DO NOT TEAR THIS PAGE FROM THE TEST.

1. $\int e^u\,du = e^u + C$

2. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, n \neq -1$

3. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

4. $\displaystyle\int \cos u\,du = \sin u + C$

5. $\displaystyle\int \sin u\,du = -\cos u + C$

6. $\displaystyle\int \sec^2 u\,du = \tan u + C$

7. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

8. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

9. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$

10. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

11. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

12. $\displaystyle\int \cot u\,du = -\ln|\csc u| + C$

13. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} + C = \arcsin\frac{u}{a} + C$

15. $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C = \frac{1}{a}\arctan\frac{u}{a} + C$

16. $\displaystyle\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C = \frac{1}{a}\text{arcsec}\left|\frac{u}{a}\right| + C$

17. $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

18. $\sin^2\theta + \cos^2\theta = 1$

19. $\tan^2\theta + 1 = \sec^2\theta$

20. $\sin(2\theta) = 2\sin\theta\,\cos\theta$

21. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

1. [8 marks] Find the exact value of each of the following integrals:

(a) [4 marks] $\displaystyle\int_0^{\pi/2} \sin^3 x \, \cos^3 x \, dx.$

Solution: let $u = \sin x$. Then $du = \cos x \, dx$, $\cos^2 x = 1 - \sin^2 x = 1 - u^2$, and

$$\begin{aligned}
\int_0^{\pi/2} \sin^3 x \, \cos^3 x \, dx &= \int_0^{\pi/2} \sin^3 x \, \cos^2 x \, \cos x \, dx \\
&= \int_0^1 u^3(1-u^2)\, du \\
&= \int_0^1 (u^3 - u^5)\, du \\
&= \left[\frac{u^4}{4} - \frac{u^6}{6}\right]_0^1 \\
&= \frac{1}{4} - \frac{1}{6} \\
&= \frac{1}{12}
\end{aligned}$$

(b) [4 marks] $\displaystyle\int_{\ln\sqrt{2}}^{\ln 2} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\, dx.$

Solution: let $u = e^{-x}$. Then $du = -e^{-x}\, dx$, and

$$\begin{aligned}
\int_{\ln\sqrt{2}}^{\ln 2} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\, dx &= \int_{1/\sqrt{2}}^{1/2} \frac{-du}{\sqrt{1-u^2}} \\
&= \int_{1/2}^{1/\sqrt{2}} \frac{du}{\sqrt{1-u^2}} \\
&= \left[\sin^{-1} u\right]_{1/2}^{1/\sqrt{2}} \\
&= \frac{\pi}{4} - \frac{\pi}{6} \\
&= \frac{\pi}{12}
\end{aligned}$$

2. [7 marks] Find $\int x^2 \tan^{-1} x \, dx$.

Solution: let $u = \tan^{-1} x$; $dv = x^2 \, dx$ and integrate by parts.

$$
\begin{aligned}
\int x^2 \tan^{-1} x \, dx &= uv - \int v \, du \\
&= \frac{x^3}{3} \tan^{-1} x - \int \frac{x^3}{3} \frac{1}{x^2+1} \, dx \\
&= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{x^2+1} \, dx \\
\text{(by long division)} \quad &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{x^2+1} \right) dx \\
&= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \left(\frac{x^2}{2} - \frac{1}{2} \ln(x^2+1) \right) + C \\
&= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(x^2+1) + C
\end{aligned}
$$

3. [8 marks] Find the exact value of $\displaystyle\int_1^6 \frac{dx}{x^2\sqrt{9x^2-4}}$.

Solution: let $3x = 2\sec\theta$; then $dx = \frac{2}{3}\sec\theta\tan\theta\,d\theta$, and for $x \geq \frac{2}{3}$ and $0 \leq \theta < \frac{\pi}{2}$,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9x^2-4}} &= \int \frac{9}{4\sec^2\theta}\frac{1}{\sqrt{4\sec^2\theta-4}}\frac{2}{3}\sec\theta\tan\theta\,d\theta \\
&= \frac{3}{4}\int \cos\theta\,d\theta \\
&= \frac{3}{4}\sin\theta + C \\
&= \frac{3}{4}\frac{\sqrt{9x^2-4}}{3x} + C, \text{ since } \cos\theta = \frac{1}{\sec\theta} = \frac{2}{3x} \text{ and } \sin\theta = \sqrt{1-\cos^2\theta} \\
&= \frac{1}{4}\frac{\sqrt{9x^2-4}}{x} + C
\end{aligned}$$

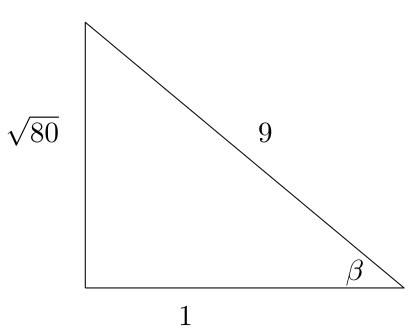
Thus

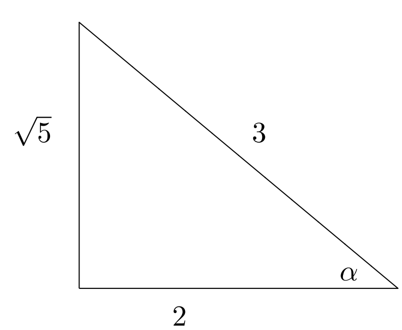
$$\int_1^6 \frac{dx}{x^2\sqrt{9x^2-4}} = \left[\frac{1}{4}\frac{\sqrt{9x^2-4}}{x}\right]_1^6 = \frac{1}{4}\frac{\sqrt{320}}{6} - \frac{\sqrt{5}}{4} = \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{4} = \frac{\sqrt{5}}{12}$$

Note: in this question it is not convenient to work with definite integrals, but it can be done:

$$\begin{aligned}
\int_1^6 \frac{dx}{x^2\sqrt{9x^2-4}} &= \int_{\sec^{-1}(3/2)}^{\sec^{-1}9} \frac{9}{4\sec^2\theta}\frac{1}{\sqrt{4\sec^2\theta-4}}\frac{2}{3}\sec\theta\tan\theta\,d\theta \\
&= \frac{3}{4}\int_{\sec^{-1}(3/2)}^{\sec^{-1}9} \cos\theta\,d\theta \\
&= \frac{3}{4}\left[\sin\theta\right]_{\sec^{-1}(3/2)}^{\sec^{-1}9} \\
&= \frac{3}{4}\sin(\sec^{-1}9) - \frac{3}{4}\sin(\sec^{-1}(3/2)) \\
&= \frac{3}{4}\frac{\sqrt{80}}{9} - \frac{3}{4}\frac{\sqrt{5}}{3}, \text{ see the two triangles below} \\
&= \frac{\sqrt{5}}{12}
\end{aligned}$$

Triangles (not to scale) with $\beta = \sec^{-1}9$ and $\alpha = \sec^{-1}(3/2)$:





4. [7 marks] Find $\int \frac{x^2}{(x-1)^3}\,dx$.

Solution 1: without using partial fractions. Let $u = x - 1$, then $x = u + 1$ and $dx = du$, so

$$\begin{aligned}
\int \frac{x^2}{(x-1)^3}\,dx &= \int \frac{(u+1)^2}{u^3}\,du \\
&= \int \frac{u^2+2u+1}{u^3}\,du \\
&= \int \left(\frac{1}{u} + \frac{2}{u^2} + \frac{1}{u^3}\right)\,du \\
&= \ln|u| - \frac{2}{u} - \frac{1}{2}\frac{1}{u^2} + C \\
&= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2}\frac{1}{(x-1)^2} + C
\end{aligned}$$

Solution 2: using partial fractions. Let

$$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$

Then

$$\begin{aligned}
x^2 &= A(x-1)^2 + B(x-1) + C \\
&= Ax^2 + (-2A+B)x + A - B + C
\end{aligned}$$

So $A = 1, B = 2, C = 1$; consequently

$$\begin{aligned}
\int \frac{x^2}{(x-1)^3}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)^3}\,dx \\
&= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2}\frac{1}{(x-1)^2} + K
\end{aligned}$$

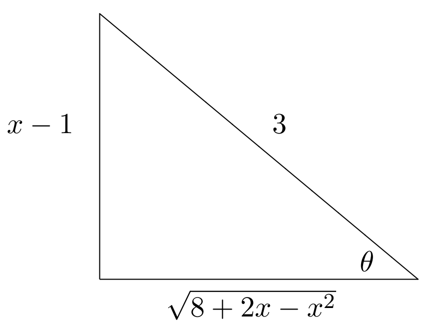
5. [8 marks] Find $\int \frac{x+1}{\sqrt{8+2x-x^2}}\,dx$.

Solution: complete the square and use a trigonometric substitution.

$$8+2x-x^2 = 9-1+2x-x^2 = 9-(x-1)^2$$

Let $x-1 = 3\sin\theta$, then $x = 1+3\sin\theta$ and

$$\begin{aligned}
\int \frac{x+1}{\sqrt{8+2x-x^2}}\,dx &= \int \frac{2+3\sin\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta \\
&= \int (2+3\sin\theta)\,d\theta \\
&= 2\theta - 3\cos\theta + C \\
&= 2\sin^{-1}\left(\frac{x-1}{3}\right) - 3\frac{\sqrt{8+2x-x^2}}{3} + C \\
&= 2\sin^{-1}\left(\frac{x-1}{3}\right) - \sqrt{8+2x-x^2} + C,
\end{aligned}$$



where we used the triangle to the left with

$$\sin\theta = \frac{x-1}{3}$$

to get

$$\cos\theta = \frac{\sqrt{8+2x-x^2}}{3}.$$

Alternate Solution: after completing the square, let $u = x-1$. Then

$$\begin{aligned}
\int \frac{x+1}{\sqrt{8+2x-x^2}}\,dx &= \int \frac{x-1+2}{\sqrt{9-(x-1)^2}}\,dx \\
&= \int \frac{u+2}{\sqrt{9-u^2}}\,du \\
&= \int \frac{u}{\sqrt{9-u^2}}\,du + \int \frac{2}{\sqrt{9-u^2}}\,du \\
&= -\sqrt{9-u^2} + 2\sin^{-1}\left(\frac{u}{3}\right) + C \\
&= -\sqrt{9-(x-1)^2} + 2\sin^{-1}\left(\frac{x-1}{3}\right) + C \\
&= -\sqrt{8+2x-x^2} + 2\sin^{-1}\left(\frac{x-1}{3}\right) + C
\end{aligned}$$

6. [7 marks] Find the value of the following improper integrals.

(a) [3 marks] $\displaystyle\int_1^\infty e^{-2x}\,dx$.

Solution:

$$\begin{aligned}
\int_1^\infty e^{-2x}\,dx &= \lim_{b\to\infty}\int_1^b e^{-2x}\,dx \\
&= \lim_{b\to\infty}\left[\frac{e^{-2x}}{-2}\right]_1^b \\
&= \lim_{b\to\infty}\left(-\frac{1}{2e^{2b}}\right)+\frac{1}{2e^2} \\
&= 0+\frac{1}{2e^2} \\
&= \frac{1}{2e^2}
\end{aligned}$$

(b) [4 marks] $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)}$.

Solution: using a substitution and the appropriate intervals of integration. Let $u=\sqrt{x}$; then $0\le x<\infty \Rightarrow 0\le\sqrt{x}=u<\infty$, whence

$$\begin{aligned}
\int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)} &= \int_0^\infty \frac{2\,du}{u^2+1} \\
&= \lim_{b\to\infty}\left[2\tan^{-1}u\right]_0^b \\
&= \lim_{b\to\infty} 2\tan^{-1}b-2\tan^{-1}0 \\
&= 2\left(\frac{\pi}{2}\right)-0 \\
&= \pi
\end{aligned}$$

7. [7 marks] Find $\int e^{-x} \sin(2x)\, dx$.

Solution: use integration by parts, twice. First let $u = e^{-x}$ and $dv = \sin(2x)dx$. Then

$$\begin{aligned}
\int e^{-x} \sin(2x)\, dx &= uv - \int v\, du \\
&= -\frac{e^{-x}\cos(2x)}{2} - \frac{1}{2}\int (-e^{-x})(-\cos(2x))\, dx \\
&= -\frac{e^{-x}\cos(2x)}{2} - \frac{1}{2}\int e^{-x}\cos(2x)\, dx \\
\text{(let } s = e^{-x}; dt = \cos(2x)dx) &= -\frac{e^{-x}\cos(2x)}{2} - \frac{1}{2}\left[st - \int t\, ds\right] \\
&= -\frac{e^{-x}\cos(2x)}{2} - \frac{1}{2}\frac{e^{-x}\sin(2x)}{2} + \frac{1}{2}\int -\frac{1}{2}e^{-x}\sin(2x)\, dx \\
&= -\frac{e^{-x}\cos(2x)}{2} - \frac{e^{-x}\sin(2x)}{4} - \frac{1}{4}\int e^{-x}\sin(2x)\, dx \\
\Rightarrow \frac{5}{4}\int e^{-x}\sin(2x)\, dx &= -\frac{e^{-x}\cos(2x)}{2} - \frac{e^{-x}\sin(2x)}{4} + C \\
\Rightarrow \int e^{-x}\sin(2x)\, dx &= -\frac{2e^{-x}\cos(2x)}{5} - \frac{e^{-x}\sin(2x)}{5} + C
\end{aligned}$$

8. [8 marks] Find $\displaystyle\int \frac{dx}{x^{5/3} - x^{1/3}}$.

Solution: let $x = u^3$; then $dx = 3u^2\,du$, $x^{5/3} = u^5$ and $x^{1/3} = u$. So

$$\int \frac{dx}{x^{5/3} - x^{1/3}} = \int \frac{3u^2\,du}{u^5 - u} = \int \frac{3u\,du}{u^4 - 1} = \frac{3}{2}\int \frac{2u\,du}{(u^2+1)(u^2-1)}.$$

Now let $v = u^2$, so $dv = 2u\,du$. Then

$$\begin{aligned}
\frac{3}{2}\int \frac{2u\,du}{(u^2+1)(u^2-1)} &= \frac{3}{2}\int \frac{dv}{(v+1)(v-1)} \\
&= \frac{3}{2}\int \left(\frac{1/2}{v-1} - \frac{1/2}{v+1}\right) dv, \text{ by partial fractions} \\
&= \frac{3}{2}\left(\frac{1}{2}\ln|v-1| - \frac{1}{2}\ln|v+1|\right) + C \\
&= \frac{3}{4}\ln\left|\frac{v-1}{v+1}\right| + C \\
&= \frac{3}{4}\ln\left|\frac{u^2-1}{u^2+1}\right| + C, \text{ since } v = u^2 \\
&= \frac{3}{4}\ln\left|\frac{x^{2/3}-1}{x^{2/3}+1}\right| + C, \text{ since } x = u^{1/3}
\end{aligned}$$

Alternate forms of the answer:

$$\frac{3}{4}\ln|x^{2/3}-1| - \frac{3}{4}\ln|x^{2/3}+1| + C,$$

or

$$\frac{3}{4}\ln|x^{1/3}-1| + \frac{3}{4}\ln|x^{1/3}+1| - \frac{3}{4}\ln|x^{2/3}+1| + C.$$

This last one would be the answer if you used partial fractions as follows:

$$\int \frac{3u\,du}{u^4-1} = \frac{3}{4}\int \frac{du}{u-1} + \frac{3}{4}\int \frac{du}{u+1} - \frac{3}{2}\int \frac{u\,du}{u^2+1} = \frac{3}{4}\int \frac{du}{u-1} + \frac{3}{4}\int \frac{du}{u+1} - \frac{3}{4}\int \frac{2u\,du}{u^2+1},$$

and $u = x^{1/3}$.

Alternate Solution: let $u = x^{2/3}$, then $du = \dfrac{2}{3}\dfrac{dx}{x^{1/3}}$, so

$$\int \frac{dx}{x^{5/3} - x^{1/3}} = \int \frac{dx}{x^{1/3}(x^{4/3}-1)} = \frac{3}{2}\int \frac{du}{u^2-1} = \frac{3}{2}\int \left(\frac{1/2}{u-1} - \frac{1/2}{u+1}\right) du = \frac{3}{4}\ln\left|\frac{u-1}{u+1}\right| + C$$

A slightly less elegant version of this approach, letting $u = x^{4/3}$, will also work.