MAT187 - Calculus II - Winter 2016

Term Test 1 - February 2, 2016

Time allotted: 90 minutes. Aids permitted: None.

Total marks: 50

SOLUTIONS

Full Name: ______________________ (Last) ______________________ (First)

Student Number: ______________________

Email: ______________________@mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection
- This test contains 12 pages and a detached **formula sheet**. Make sure you have all of them. DO NOT DETACH ANY PAGE.
- You can use page 12 to complete questions (**mark clearly** which questions you are answering on page 12).
- Calculators, cellphones, or any other electronic gadgets are not allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.
- DO NOT start the test until instructed to do so.

GOOD LUCK!

PART I . Write the final answer answer only. (10 marks)

1. **(2 mark)** Calculate $\int_1^e x^2 \ln(x)\, dx =$ $\dfrac{2e^3+1}{9}$.

2. **(1 mark)** To calculate the integral $\int \frac{1}{\sqrt{4x^2-9}}\, dx$, propose a substitution that simplifies the problem. **(circle one choice)**

(a) $x = \frac{2}{3}\cos u$ (e) $x = \frac{2}{3}\tan u$ (i) $x = \frac{2}{3}\cosh u$

(b) $x = \frac{3}{2}\cos u$ (f) $x = \frac{3}{2}\tan u$ (j) $x = \frac{3}{2}\cosh u$

(c) $x = \frac{2}{3}\sin u$ (g) $x = \frac{2}{3}\sec u$ (k) $x = \frac{2}{3}\sinh u$

(d) $x = \frac{3}{2}\sin u$ (h) $x = \frac{3}{2}\sec u$ (l) $x = \frac{3}{2}\sinh u$

3. **(2 marks)** Suppose that approximating the integral $\int_a^b f(x)\, dx$ using the Trapezoidal Rule with $n = 10$ intervals yields the error estimate $|E_T| \leqslant \frac{b-a}{12}(\Delta x)^2 \max\limits_{x\in[a,b]} |f''(x)| \leqslant \frac{5}{2}$.

What value of n should we use to make sure the error is $|E_T| \leqslant \frac{1}{1000}$?

$n \geqslant$ 500

4. **(1 mark)** Consider a continuous function $f(x)$ such that $f(x) > 1$ for all $x \in [0,1]$, and consider the area bounded by the graphs $y = f(x)$, $y = 1$, $x = 0$, and $x = 1$. Then revolve that area around the axis $y = 1$ to obtain a solid of revolution.

Its volume is given by the integral formula $V =$ $\pi \int_0^1 [f(x) - 1]^2\, dx$.

5. **(2 marks)** Consider the rational function $\dfrac{x^9}{(x^2+6x+8)^2(x^2+2x+3)^3}$. When using **partial fractions**, we can write this function as a sum of the following terms (**circle all that apply**):

(a) $\dfrac{A}{x+4}$ (d) $\dfrac{D}{x+2}$ (g) $\dfrac{Gx+H}{x^2+6x+8}$ (j) $\dfrac{Mx+N}{x^2+2x+3}$

(b) $\dfrac{B}{(x+4)^2}$ (e) $\dfrac{E}{(x+2)^2}$ (h) $\dfrac{Ix+J}{(x^2+6x+8)^2}$ (k) $\dfrac{Ox+P}{(x^2+2x+3)^2}$

(c) $\dfrac{C}{(x+4)^3}$ (f) $\dfrac{F}{(x+2)^3}$ (i) $\dfrac{Kx+L}{(x^2+6x+8)^3}$ (l) $\dfrac{Qx+R}{(x^2+2x+3)^3}$

For questions **6.** and **7.**, consider a function $f(x)$ which is continuous for all $x \in \mathbb{R}$ and $0 \leqslant f(x) \leqslant \dfrac{1}{x^2}$ for $x \in (1, \infty)$.

6. **(1 mark)** Then **(circle one choice)**

(a) $\displaystyle\int_{-100}^{+\infty} f(x)\,dx$ converges.

(b) $\displaystyle\int_{-100}^{+\infty} f(x)\,dx$ diverges.

(c) We cannot tell whether $\displaystyle\int_{-100}^{+\infty} f(x)\,dx$ converges or diverges.

7. **(1 mark)** Then **(circle one choice)**

(a) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ converges.

(b) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ diverges.

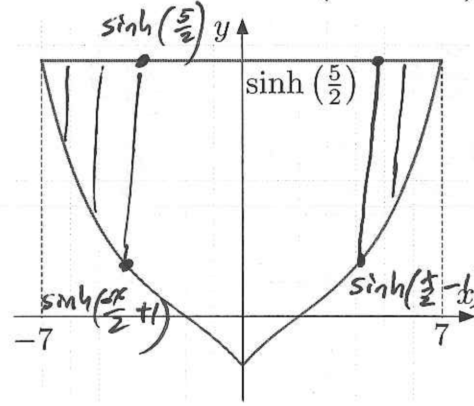
(c) We cannot tell whether $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ converges or diverges.

PART II **Justify** your answer for all questions in this part.

8. A ship has a hull whose cross section has the shape **(10 marks)**

$$y = \sinh\left(\frac{x}{2} - 1\right) \qquad \text{for } x \in [0, 7],$$

and it is symmetric with respect to the y-axis, as in the figure on the right.



Mr. Ng works at the shipyard and his task is to find the area of this cross section.

(a) **(5 marks)** Find an integral formula for the area of the cross section. (do not calculate the integral)

$$A = 2\int_0^7 \sinh\left(\tfrac{5}{2}\right) - \sinh\left(\tfrac{x}{2} - 1\right) dx$$

OR

$$A = \int_{-7}^{0} \sinh\left(\tfrac{5}{2}\right) - \sinh\left(1 - \tfrac{x}{2}\right) dx + \int_0^7 \sinh\left(\tfrac{5}{2}\right) - \sinh\left(\tfrac{x}{2} - 1\right) dx$$

$A =$

(b) **(5 marks)** Mr. Ng didn't do very well in MAT187, so he doesn't know how to integrate this function. Instead he decides to approximate the integral using the midpoint rule.

What are the possible values of Δx that Mr. Ng can use to make sure that the error of his approximation is at most $\frac{7}{48}$? You can express Δx in terms of $\alpha = \sinh\left(\frac{5}{2}\right)$.

$$|E_M| \le \frac{b-a}{24}(\Delta x)^2 \max_{x \in [a,b]} |f''(x)|$$

So: $b = 7$, $a = 0$, $\Delta x = \frac{7-(0)}{n} = \frac{7}{n}$

$$f(x) = \sinh\left(\tfrac{5}{2}\right) - \sinh\left(\tfrac{x}{2}-1\right) = \sinh\left(\tfrac{5}{2}\right) - \frac{e^{\frac{x}{2}-1} - e^{1-\frac{x}{2}}}{2}$$

$$f'(x) = -\tfrac{1}{2}\cosh\left(\tfrac{x}{2}-1\right) = -\frac{e^{\frac{x}{2}-1} + e^{1-\frac{x}{2}}}{4}$$

$$f''(x) = -\tfrac{1}{4}\sinh\left(\tfrac{x}{2}-1\right) = -\frac{e^{\frac{x}{2}-1} - e^{1-\frac{x}{2}}}{8}$$

On $[0,7]$, $|f''(x)| \le \frac{1}{4}\sinh\left(\frac{5}{2}\right)$

[This can be seen from the diagram in (a)]

Since this only gives half of the area, we need to double it for the overall error!

$$\underset{\text{on } [-7,7]}{|E_M|} \le 2 \cdot \frac{7}{24} \cdot \left(\frac{7}{n}\right)^2 \cdot \frac{1}{4}\sinh\left(\tfrac{5}{2}\right) \le \frac{7}{48}$$

$$\frac{7^3 \sinh\left(\frac{5}{2}\right)}{48 n^2} \le \frac{7}{48}$$

$$n^2 \ge 49 \sinh\left(\tfrac{5}{2}\right)$$

or

$$|E_M| \le 2 \cdot \frac{7}{24} \cdot (\Delta x)^2 \frac{1}{4}\sinh\left(\tfrac{5}{2}\right) \le \frac{7}{48}$$

$$\frac{7}{48}(\Delta x)^2 \sinh(5/2) \le \frac{7}{48}$$

$$(\Delta x)^2 \le \frac{1}{\sinh(5/2)}$$

Answer: $n \ge 7\sqrt{\sinh\left(\frac{5}{2}\right)}$ or $\Delta x \le \frac{1}{\sqrt{\alpha}}$

9. Consider the integral **(10 marks)**

$$\int_0^{\infty} \frac{\sqrt{x}-1}{(x^{\frac{3}{2}}+1)\sqrt{x}}\,dx.$$

(a) **(4 marks)** This is an improper integral. Break the integral into appropriate limits.

The integral is improper because (a) it has an infinite discontinuity at $x=0$ and (b) its domain is unbounded. We need a limit to each of those:

$$\lim_{a\to 0^+}\int_a^1 \frac{\sqrt{x}-1}{(x^{\frac{3}{2}}+1)\sqrt{x}}\,dx + \lim_{b\to\infty}\int_1^b \frac{\sqrt{x}-1}{(x^{\frac{3}{2}}+1)\sqrt{x}}\,dx$$

[Note that we can use any positive constant for the break.]

(b) **(4 marks)** Use the substitution $x=u^2$ and calculate the integrals.

Hint. $x^3+1=(x+1)(x^2-x+1)$. $\hookrightarrow dx = 2u\,du$ and $u=\sqrt{x}$

$$\int \frac{\sqrt{x}-1}{(x^{\frac{3}{2}}+1)\sqrt{x}}\,dx = \int \frac{(u-1)\cdot 2\cancel{u}\,du}{(u^3+1)\cdot \cancel{u}} = \int \frac{u-1}{(u+1)(u^2-u+1)}\,du$$

Let $\frac{u-1}{(u+1)(u^2-u+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2-u+1}$

$$= \frac{(A+B)u^2+(-A+B+C)u+(A+C)}{(u+1)(u^2-u+1)}$$

$\therefore$ $A+B=0$

$-A+B+C=1$

$A+C=-1$

so, $A=-\frac{2}{3}$, $B=\frac{2}{3}$, $C=-\frac{1}{3}$

$\therefore$ Integral $= -\frac{2}{3}\int \frac{1}{u+1}\,du + \frac{1}{3}\int \frac{2u-1}{u^2-u+1}\,du$

$= -\frac{2}{3}\ln|u+1| + \frac{1}{3}\ln(u^2-u+1) + C$

Returning to the definite integral:

$$\int_a^1 \frac{\sqrt{x}-1}{(x^{\frac{3}{2}}+1)\sqrt{x}}\,dx = \left[-\frac{2}{3}\ln(2) + \frac{1}{3}\ln(1)\right] - \left[-\frac{2}{3}\ln(\sqrt{a}+1) + \frac{1}{3}\ln(a-\sqrt{a}+1)\right]$$

$$= -\frac{2}{3}\ln(2) + \frac{2}{3}\ln(\sqrt{a}+1) - \frac{1}{3}\ln(a-\sqrt{a}+1)$$

$$\int_1^b \frac{\sqrt{x}-1}{(x^{\frac{3}{2}}+1)\sqrt{x}}\,dx = \left[-\frac{2}{3}\ln(\sqrt{b}+1) + \frac{1}{3}\ln(b-\sqrt{b}+1)\right] + \frac{2}{3}\ln(2)$$

(c) **(2 marks)** Conclude whether the integral converges or diverges. If it converges, write the limit.

$$\lim_{a\to 0^+}\left[-\frac{2}{3}\ln(2) + \frac{2}{3}\ln(\sqrt{a}+1) - \frac{1}{3}\ln(a-\sqrt{a}+1)\right]$$

$$= -\frac{2}{3}\ln(2) + \frac{2}{3}\ln(1) - \frac{1}{3}\ln(1) = -\frac{2}{3}\ln(2)$$

$$\lim_{b\to\infty}\left[-\frac{2}{3}\ln(\sqrt{b}+1) + \frac{1}{3}\ln(b-\sqrt{b}+1)\right] + \frac{2}{3}\ln(2)$$

$$= \lim_{b\to\infty} \frac{1}{3}\ln\left[\frac{b-\sqrt{b}+1}{(\sqrt{b}+1)^2}\right] + \frac{2}{3}\ln(2)$$

$$= \lim_{b\to\infty} \frac{1}{3}\ln\left[\frac{1-\frac{1}{\sqrt{b}}+\frac{1}{b}}{1+\frac{2}{\sqrt{b}}+\frac{1}{b}}\right] + \frac{2}{3}\ln(2) = \frac{2}{3}\ln(2)$$

$\therefore \int_0^\infty \frac{\sqrt{x}-1}{(x^{\frac{3}{2}}+1)\sqrt{x}}\,dx$ converges to 0.

10. In this exercise, we want to figure out how much work is carried out to take a Falcon-9 rocket to low Earth orbit. **(10 marks)**

The empty rocket has a mass m_r.

The gravitational force is $F = \frac{GMm}{d^2}$, where M is the mass of the Earth, m is the **total mass of the rocket**, d is the distance between the rocket and the Earth's centre, and G is the universal gravitational constant. The radius of the Earth is r_E.

As the rocket gets higher, the fuel is expended, such that the mass of the fuel in the rocket is

$$m_f(y) = \frac{10m_r}{1+y} \qquad \text{where } y \text{ is the altitude of the rocket above the surface of the Earth.}$$

(a) **(3 marks)** Obtain a formula for the **gravitational force**.

G and M are constants.

$m(y) = m_r + m_f(y) = m_r + \frac{10 m_r}{1+y}$

$d(y) = y + r_E$

$$\boxed{F(y) = GM \cdot \frac{m_r + \frac{10 m_r}{1+y}}{(y+r_E)^2}}$$

(b) **(4 marks)** Use partial fractions to write the **gravitational force** as a sum of irreducible fractions.

$$F(y) = GMm_r \frac{y+11}{(y+1)(y+r_E)^2}$$

$$= GMm_r \left[\frac{A}{y+1} + \frac{B}{y+r_E} + \frac{C}{(y+r_E)^2}\right]$$

So, $$\frac{y+11}{(y+1)(y+r_E)^2} = \frac{A(y+r_E)^2 + B(y+1)(y+r_E) + C(y+1)}{(y+1)(y+r_E)^2}$$

Let's use the Heaviside method:

For $y=-1$, $10 = A(-1+r_E)^2 + 0 + 0$

$$\therefore A = \frac{10}{(-1+r_E)^2}$$

For $y = -r_E$, $11 - r_E = 0 + 0 + C(1 - r_E)$

$$\therefore C = \frac{11 - r_E}{1 - r_E}$$

For $y = 0$, $11 = \frac{10 r_E^2}{(-1 + r_E)^2} + B \cdot 1 \cdot r_E + \frac{11 - r_E}{1 - r_E} \cdot 1$

$$\therefore B = \frac{1}{r_E}\left[11 - \frac{10 r_E^2}{(r_E - 1)^2} - \frac{11 - r_E}{1 - r_E}\right]$$

$$\therefore F(y) = GMm_r \left[\frac{\left(\frac{10}{(r_E - 1)^2}\right)}{y + 1} + \frac{\frac{1}{r_E}\left[11 - \frac{10 r_E^2}{(r_E - 1)^2} - \frac{11 - r_E}{1 - r_E}\right]}{y + r_E} + \frac{\left(\frac{11 - r_E}{1 - r_E}\right)}{(y + r_E)^2} \right]$$

(c) **(2 marks)** Obtain an **integral** formula for the amount of work required to lift this rocket to an altitude of A metres above the surface of the Earth.

$$W = \int_0^A F(y)\, dy$$

$$= \int_0^A GMm_r \,[\text{The function above}]\, dy$$

(d) **(1 marks)** Calculate the integral obtained above.

$$= GMm_r \Bigg[(\ln(A+1)) \cdot \frac{10}{(r_E - 1)^2} + (\ln(A + r_E)) \cdot \frac{1}{r_E}\left[11 - \frac{10 r_E^2}{(r_E - 1)^2} - \frac{11 - r_E}{1 - r_E}\right]$$

$$+ \frac{-1}{A + r_E} \cdot \frac{11 - r_E}{1 - r_E} - \ln(1) \cdot \frac{10}{(r_E - 1)^2} - \ln(r_E) \cdot \frac{1}{r_E}\Big[11 - \dots$$

$$- \frac{10 r_E^2}{(r_E - 1)^2} - \frac{11 - r_E}{1 - r_E}\Big] + \frac{-1}{r_E} \cdot \frac{11 - r_E}{1 - r_E} \Bigg]$$

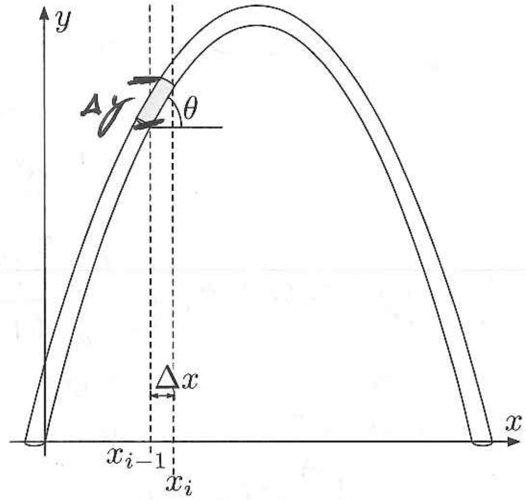
11. Consider a circular tube of constant radius r (in metres), that is bent into the shape of an arch, as shown in the figure. **(10 marks)**

The formula for the shape of the arch is

$$y = x(1-x) \quad \text{for } x \in [0,1],$$

where the units of x are in metres.

The mass density is $\rho(x,y) = x(1-x)$ kg/m^3.
Find an integral formula for the mass of the arch.
(Do not calculate the integral)



Note. There is more than one way to this. You may use the formula $\sec(\arctan x) = \sqrt{1+x^2}$.

The mass for the small section above is

m = Volume · density

$\approx$ length ·(cross-section area) · density since the tube is nearly cylindrical in small sections

$$\approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \cdot \pi r^2 \cdot x(1-x)$$

$$\approx \Delta x \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}\, \pi r^2 \cdot x(1-x)$$

$$\approx \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \pi r^2\, x(1-x)\, \Delta x$$

or

length $= \Delta x \sec\theta$

$\tan\theta = y'(x)$

$\Rightarrow \sec\theta = \sqrt{1 + (y'(x))^2}$

$$\Rightarrow m \approx \sqrt{1+(y'(x))^2}\, \pi r^2 x(1-x)\Delta x$$

To be used for the answer for question **11.**

$\therefore$ The total mass is

$$M = \lim_{\substack{\Delta x \to 0 \\ \text{or } n\to\infty}} \sum_{i=1}^{n} \pi r^2 \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2} \cdot x(1-x)\,\Delta x$$

$$= \int_0^1 \pi r^2 \sqrt{1+(y')^2}\; x(1-x)\,dx$$

Now, $y = x(1-x) = x - x^2$

$\therefore y' = 1-2x$

$1+(y')^2 = 2 - 4x + 4x^2$

$$= \int_0^1 \pi r^2 \cdot x(1-x)\sqrt{2-4x+4x^2}\,dx$$

Answer:

USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

The end.