University of Toronto

Solutions to MAT 187H1S TERM TEST

of **Thursday, February 2, 2012**

Duration: 100 minutes

General Comments:

1. Generally the results on this test were very good. Anybody who failed this test is in deep trouble, since the course material only gets harder!

2. Too many students are still making inexcusable errors, such as

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} \text{ or } \int \frac{dx}{f(x)} = \ln|f(x)| + C.$$

   And there are still students using bad notation–confusing $=$ and $\Rightarrow$, or dropping dx from the integral, etc.–and consequently throwing away marks.

3. In Questions 1, 2 and 6 you should simplify your answers, eg $\ln 4 - \ln 2 = \ln 2$.

4. Indefinite integrals require a constant of integration; definite integrals do not.

5. If in Questions 1, 2 or 6 you make a substitution in the definite integral, then you must change the limits of integration as you change the variable. Theorem 5.9.1 on page 391 of the textbook spells it out:

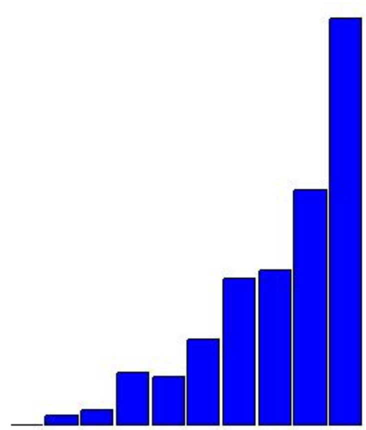
$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du,$$

   where the substitution is $u = g(x)$. Doing this will actually save you work.

6. The whole point of the Question 5 is how to handle the minus sign in $\sqrt{5+4x-x^2}$. Since $5+4x-x^2 \geq 0 \Rightarrow -3 \leq x-2 \leq 3$, letting $x-2 = 3\sec\theta$ or $3\tan\theta$ makes no sense at all.

Breakdown of Results: 482 students wrote this test. The marks ranged from 15% to 100%, and the average was 76.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	35.3%
A	55.6%	80-89%	20.3%
B	13.5%	70-79%	13.5%
C	12.7%	60-69%	12.7%
D	7.5%	50-59%	7.5%
F	10.7%	40-49%	4.1%
		30-39%	4.6%
		20-29%	1.2%
		10-19%	0.8%
		0-9%	0.0%



1. [9 marks] Find the exact values of the following integrals:

(a) [4 marks] $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}\, e^{\sqrt{x}}}$

Solution: let $u = \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}\, dx$ and

$$\int_4^9 \frac{dx}{\sqrt{x}\, e^{\sqrt{x}}} = 2\int_1^3 e^{-u}\, du = 2\left[-e^{-u}\right]_1^3 = \frac{2}{e^2} - \frac{2}{e^3}$$

(b) [5 marks] $\displaystyle\int_0^{\pi/4} \tan^3 x \sec^4 x\, dx$

Solution: let $u = \tan x$. Then $du = \sec^2 x\, dx$, and

$$\begin{aligned}
\int_0^{\pi/4} \tan^3 x \sec^4 x\, dx &= \int_0^{\pi/4} \tan^3 x \sec^2 x \sec^2 x\, dx \\
&= \int_0^1 u^3(1+u^2)\, du \\
&= \int_0^1 \left(u^3 + u^5\right)\, du \\
&= \left[\frac{u^4}{4} + \frac{u^6}{6}\right]_0^1 \\
&= \frac{1}{4} + \frac{1}{6} \\
&= \frac{5}{12}
\end{aligned}$$

2. [8 marks] Find the exact value of $\displaystyle\int_1^3 \sqrt{x}\,\tan^{-1}\sqrt{x}\,dx$

Solution: let $u = \tan^{-1}\sqrt{x}$; $dv = \sqrt{x}\,dx$ and integrate by parts.

$$\begin{aligned}
\int_1^3 \sqrt{x}\,\tan^{-1}\sqrt{x}\,dx &= [uv]_1^3 - \int_1^3 v\,du \\
&= \left[\frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\right]_1^3 - \int_1^3 \frac{2}{3}x^{3/2}\frac{1}{x+1}\frac{1}{2\sqrt{x}}\,dx \\
&= \left[\frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\right]_1^3 - \frac{1}{3}\int_1^3 \frac{x}{1+x}\,dx \\
&= \left[\frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\right]_1^3 - \frac{1}{3}\int_1^3 \left(1-\frac{1}{1+x}\right)dx \\
&= \left[\frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\right]_1^3 - \frac{1}{3}\left[x-\ln(1+x)\right]_1^3 \\
&= 2\sqrt{3}\tan^{-1}\sqrt{3} - \frac{2}{3}\tan^{-1}1 - \frac{1}{3}(3-\ln 4 - 1 + \ln 2) \\
&= \frac{2\pi}{\sqrt{3}} - \frac{\pi}{6} - \frac{2}{3} + \frac{\ln 2}{3}
\end{aligned}$$

Alternate Solution: let $x = t^2$. Then $dx = 2t\,dt$ and

$$\int_1^3 \sqrt{x}\,\tan^{-1}\sqrt{x}\,dx = \int_1^{\sqrt{3}} (t\tan^{-1}t)\,(2t)\,dt = 2\int_1^{\sqrt{3}} t^2\tan^{-1}t\,dt.$$

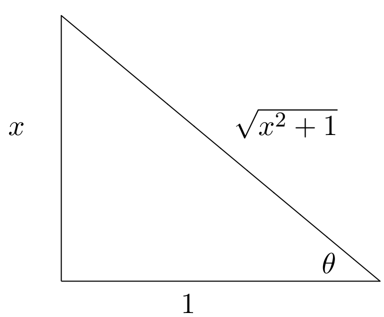
Now use integration by parts with $u = \tan t$ and $dv = t^2\,dt$. Then

$$\begin{aligned}
2\int_1^{\sqrt{3}} t^2\tan^{-1}t\,dt &= 2[uv]_1^{\sqrt{3}} - 2\int_1^{\sqrt{3}} v\,du \\
&= 2\left[\frac{t^3}{3}\tan^{-1}t\right]_1^{\sqrt{3}} - 2\int \frac{t^3}{3}\frac{dt}{t^2+1} \\
&= 2\left[\frac{t^3}{3}\tan^{-1}t\right]_1^{\sqrt{3}} - \frac{2}{3}\int \frac{t^3}{t^2+1}\,dt \\
&= 2\left[\frac{t^3}{3}\tan^{-1}t\right]_1^{\sqrt{3}} - \frac{2}{3}\int \left(t - \frac{t}{t^2+1}\right)dt \\
&= 2\left[\frac{t^3}{3}\tan^{-1}t\right]_1^{\sqrt{3}} - \frac{2}{3}\left[\frac{t^2}{2} - \frac{1}{2}\ln(t^2+1)\right]_1^{\sqrt{3}} \\
&= \frac{2\pi}{\sqrt{3}} - \frac{\pi}{6} - \frac{2}{3} + \frac{\ln 2}{3}
\end{aligned}$$

3. [9 marks] Find $\int \frac{dx}{x^4\sqrt{x^2+1}}$

Solution: let $x = \tan\theta$; then $dx = \sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{dx}{x^4\sqrt{x^2+1}} &= \int \frac{\sec^2\theta\, d\theta}{\tan^4\theta\,\sqrt{\tan^2\theta+1}} \\
&= \int \frac{\sec\theta\, d\theta}{\tan^4\theta} \\
&= \int \frac{\cos^3\theta}{\sin^4\theta}\, d\theta \\
&= \int \frac{\cos^2\theta}{\sin^4\theta}\, \cos\, d\theta \\
(\text{let } u = \sin\theta) &= \int \frac{(1-u^2)}{u^4}\, du \\
&= \int (u^{-4} - u^{-2})\, du \\
&= -\frac{1}{3u^3} + \frac{1}{u} + C \\
&= -\frac{1}{3\sin^3\theta} + \frac{1}{\sin\theta} + C
\end{aligned}$$



From the triangle,

$$\sin\theta = \frac{x}{\sqrt{x^2+1}}.$$

Thus

$$\int \frac{dx}{x^4\sqrt{x^2+1}} = -\frac{1}{3}\frac{(1+x^2)^{3/2}}{x^3} + \frac{\sqrt{1+x^2}}{x} + C$$

4. [9 marks] Find $\displaystyle\int \frac{6x^3 - 2x^2 + 2x - 1}{x^4 + x^2}\, dx$

Solution: use the method of partial fractions. Let

$$\frac{6x^3 - 2x^2 + 2x - 1}{x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}.$$

Then

$$\begin{aligned} 6x^3 - 2x^2 + 2x - 1 &= Ax(x^2+1) + B(x^2+1) + (Cx+D)(x^2 \\ &= (A+C)x^3 + (B+D)x^2 + Ax + B \end{aligned}$$

$$\Rightarrow A = 2, B = -1, C = 4, D = -1.$$

So

$$\begin{aligned} \int \frac{6x^3 - 2x^2 + 2x - 1}{x^4 + x^2}\, dx &= \int \frac{2}{x}\, dx - \int \frac{1}{x^2}\, dx + \int \frac{4x}{x^2+1}\, dx - \int \frac{dx}{x^2+1} \\ &= 2\ln|x| + \frac{1}{x} + 2\ln(x^2+1) - \tan^{-1} x + K \end{aligned}$$

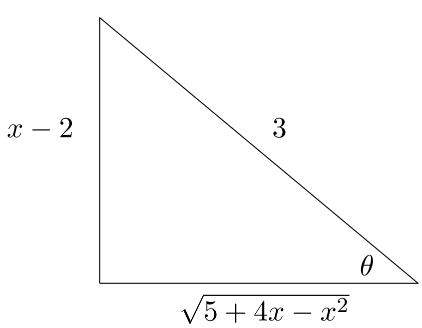
5. [9 marks] Find $\int \sqrt{5+4x-x^2}\,dx$

Solution: complete the square and use a trigonometric substitution.

$$5+4x-x^2 = 9-4+4x-x^2 = 9-(x-2)^2$$

Let $x-2 = 3\sin\theta$, then $x = 2+3\sin\theta$ and

$$\begin{aligned}
\int \sqrt{5+4x-x^2}\,dx &= \int \sqrt{9-9\sin^2\theta}\,(3\cos\theta)\,d\theta \\
&= 9\int \cos^2\theta\,d\theta \\
&= 9\int \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{9}{2}\left(\theta+\frac{\sin(2\theta)}{2}\right)+C \\
&= \frac{9}{2}\theta+\frac{9}{2}\sin\theta\cos\theta+C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\left(\frac{x-2}{3}\right)\frac{\sqrt{5+4x-x^2}}{3}+C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{(x-2)}{2}\sqrt{5+4x-x^2}+C
\end{aligned}$$



where we used the triangle to the left to get

$$\cos\theta = \frac{\sqrt{5+4x-x^2}}{3};$$

and

$$\sin\theta = \frac{x-2}{3}.$$

6. [9 marks] Determine if each of the following improper integrals converges or diverges. If it converges, find its exact value.

(a) [4 marks] $\displaystyle\int_3^\infty \frac{2\,dx}{x^2-1}$

Solution: could use a trig substitution, namely $x = \sec\theta$. Then

$$\begin{aligned}
\int_3^\infty \frac{2\,dx}{x^2-1} &= \int_{\sec^{-1}(3)}^{\pi/2} \frac{2\sec\theta\tan\theta\,d\theta}{\sec^2\theta - 1} \\
&= 2\int_{\sec^{-1}(3)}^{\pi/2} \csc\theta\,d\theta \\
&= 2\left[\ln|\csc\theta - \cot\theta|\right]_{\sec^{-1}(3)}^{\pi/2} \\
&= 2\left((\ln(1-0) - \ln\left|\frac{3}{\sqrt{8}} - \frac{1}{\sqrt{8}}\right|\right) \\
&= -2\ln(2/\sqrt{8}) = \ln 2
\end{aligned}$$

Alternate Solution: use partial fractions.

$$\int_3^\infty \frac{2\,dx}{x^2-1} = \lim_{b\to\infty}\int_3^b \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = \lim_{b\to\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_3^b = \ln 1 - \ln(2^{-1}) = \ln 2.$$

(b) [5 marks] $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x-1}}$

Solution: let $x - 1 = t^2,\ t > 0$. Then $dx = 2t\,dt$ and

$$\begin{aligned}
\int_1^\infty \frac{dx}{x\sqrt{x-1}} &= \int_0^\infty \frac{2t\,dt}{(t^2+1)\,t} \\
&= 2\int_0^\infty \frac{dt}{t^2+1} \\
&= 2\lim_{b\to\infty}\left[\tan^{-1} t\right]_0^b \\
&= 2\lim_{b\to\infty}\tan^{-1} b - \tan^{-1} 0 \\
&= 2\left(\frac{\pi}{2}\right) - 0 \\
&= \pi
\end{aligned}$$

Alternate Solution: let $u = \sec^{-1}\sqrt{x}$. Then $du = \dfrac{1}{\sqrt{x}\sqrt{x-1}}\dfrac{1}{2\sqrt{x}}\,dx = \dfrac{dx}{2x\sqrt{x-1}}$ and

$$\int_1^\infty \frac{dx}{x\sqrt{x-1}} = \int_0^{\pi/2} 2\,du = \pi.$$

7. [7 marks] Find $\int \sin(\ln x)\,dx$

Solution: use integration by parts, twice. First let $u = \sin(\ln x)$ and $dv = dx$. Then

$$\begin{aligned}
\int \sin(\ln x)\,dx &= uv - \int v\,du \\
&= x\sin(\ln x) - \int x\left(\frac{\cos(\ln x)}{x}\right)\,dx \\
&= x\sin(\ln x) - \int \cos(\ln x)\,dx \\
(\text{let } s = \cos(\ln x); dt = dx) \quad &= x\sin(\ln x) - [st - \int t\,ds] \\
&= x\sin(\ln x) - x\cos(\ln x) + \int x\left(\frac{-\sin(\ln x)}{x}\right)\,dx \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx \\
\Rightarrow 2\int \sin(\ln x)\,dx &= x\sin(\ln x) - x\cos(\ln x) + C \\
\Rightarrow \int \sin(\ln x)\,dx &= \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C
\end{aligned}$$

Alternate Solution: let $x = e^t$. Then $\ln x = t,\ dx = e^t\,dt$, and so

$$\int \sin(\ln x)\,dx = \int e^t\ \sin t\,dt.$$

This can be done by using integration by parts, twice, as illustrated in Example 5, page 494 of the textbook, albeit applied to

$$\int e^t\ \cos t\,dt.$$