University of Toronto
Solutions to MAT 187H1S TERM TEST
of **Thursday, February 3, 2011**
Duration: 100 minutes

General Comments:

1. The results on this test were much better than on the corresponding test last year. Almost half the class got A. But still, almost 11% of the class failed.

2. Many students 'simplified' the integrand in Question 6(b) by writing

$$\frac{1}{\sqrt{x}(x+1)} = \frac{1}{x\sqrt{x}} + \frac{1}{\sqrt{x}},$$

   which is not only wrong but inexcusable, and resulted in a grade of 0 for the question.

3. Indefinite integrals require a constant of integration; definite integrals do not.

4. Questions 1(b) and 2 require you to make a substituion into a definite integral. The correct way to do this is to change the limits of integration as you change the variable. Theorem 5.9.1 on page 391 of the textbook spells it out:

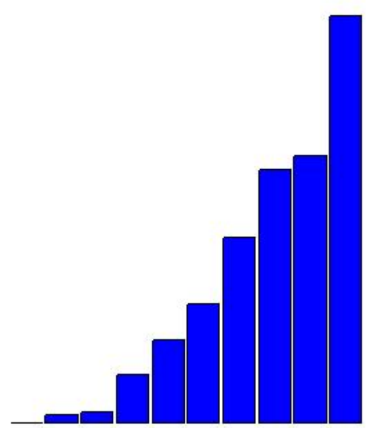
$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du,$$

   where the substitution is $u = g(x)$. Doing this will actually save you work.

5. In Question 5, $2x - x^2 \geq 0 \Rightarrow -1 \leq x - 1 \leq 1$, so letting $x - 1 = \sec\theta$ makes no sense at all.

Breakdown of Results: 486 students wrote this test. The marks ranged from 10% to 100%, and the average was 75.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	29.4%
A	48.7%	80-89%	19.3%
B	18.3%	70-79%	18.3%
C	13.4%	60-69%	13.4%
D	8.7%	50-59%	8.7%
F	10.9%	40-49%	6.0%
		30-39%	3.5%
		20-29%	0.8%
		10-19%	0.6%
		0-9%	0.0%



Formulas you may find useful. Do not tear this page from the test.

1. $\int e^u \, du = e^u + C$
2. $\displaystyle\int u^n \, du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
3. $\displaystyle\int \frac{1}{u} \, du = \ln|u| + C$
4. $\displaystyle\int \cos u \, du = \sin u + C$
5. $\displaystyle\int \sin u \, du = -\cos u + C$
6. $\displaystyle\int \sec^2 u \, du = \tan u + C$
7. $\displaystyle\int \sec u \tan u \, du = \sec u + C$
8. $\displaystyle\int \csc^2 u \, du = -\cot u + C$
9. $\displaystyle\int \csc u \cot u \, du = -\csc u + C$
10. $\displaystyle\int \tan u \, du = \ln|\sec u| + C$
11. $\displaystyle\int \sec u \, du = \ln|\sec u + \tan u| + C$
12. $\displaystyle\int \cot u \, du = -\ln|\csc u| + C$
13. $\displaystyle\int \csc u \, du = -\ln|\csc u + \cot u| + C$
14. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1} \frac{u}{a} + C = \arcsin \frac{u}{a} + C$
15. $\displaystyle\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C = \frac{1}{a} \arctan \frac{u}{a} + C$
16. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| + C = \frac{1}{a} \text{arcsec} \left|\frac{u}{a}\right| + C$
17. $\displaystyle\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$
18. $\sin^2 \theta + \cos^2 \theta = 1$
19. $\tan^2 \theta + 1 = \sec^2 \theta$
20. $\sin(2\theta) = 2 \sin \theta \, \cos \theta$
21. $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$

1. [9 marks] Find the following integrals:

(a) [4 marks] $\int e^{-x} \tan(e^{-x})\, dx$

Solution: let $u = e^{-x}$, then $du = -e^{-x}\, dx$ and

$$\int e^{-x} \tan(e^{-x})\, dx = -\int \tan u\, du = -\ln|\sec u| + C = -\ln|\sec(e^{-x})| + C$$

(b) [5 marks] $\int_0^{\pi/2} \sin^3 x \cos^2 x\, dx$

Solution: let $u = \cos x$. Then $du = -\sin x\, dx$, and

$$\begin{aligned}
\int_0^{\pi/2} \sin^3 x \cos^2 x\, dx &= \int_0^{\pi/2} \sin^2 x \cos^2 x \sin x\, dx \\
&= \int_0^{\pi/2} (1 - \cos^2 x) \cos^2 x \sin x\, dx \\
\text{(change limits!)} \quad &= -\int_1^0 \left(1 - u^2\right) u^2\, du \\
&= \int_0^1 \left(u^2 - u^4\right) du \\
&= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 \\
&= \frac{1}{3} - \frac{1}{5} \\
&= \frac{2}{15}
\end{aligned}$$

2. [8 marks] Find $\displaystyle\int_1^4 \sec^{-1}\sqrt{\theta}\,d\theta$

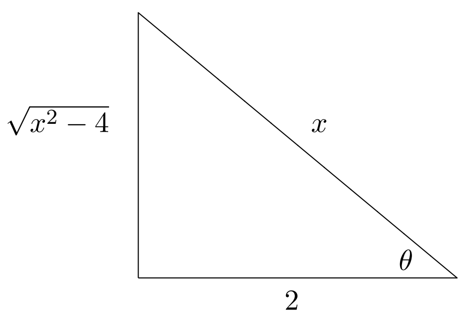
Solution: let $u = \sec^{-1}\sqrt{\theta}$ and integrate by parts.

$$\begin{aligned}
\int_1^4 \sec^{-1}\sqrt{\theta}\,d\theta &= [uv]_1^4 - \int_1^4 v\,du \\
&= \left[\theta \sec^{-1}\sqrt{\theta}\right]_1^4 - \int_1^4 \frac{\theta}{\sqrt{\theta}\sqrt{\theta-1}}\,\frac{1}{2\sqrt{\theta}}\,d\theta \\
&= \left[\theta \sec^{-1}\sqrt{\theta}\right]_1^4 - \frac{1}{2}\int_1^4 \frac{1}{\sqrt{\theta-1}}\,d\theta \\
&= \left[\theta \sec^{-1}\sqrt{\theta}\right]_1^4 - \left[\sqrt{\theta-1}\right]_1^4 \\
&= 4\sec^{-1}2 - \sec^{-1}1 - \sqrt{3} + 0 \\
&= \frac{4\pi}{3} - \sqrt{3}
\end{aligned}$$

3. [9 marks] Find $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-4}}$

Solution: let $x = 2\sec\theta$; then $dx = 2\sec\theta\tan\theta\,d\theta$ and

$$
\begin{aligned}
\int \frac{dx}{x^4\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\,d\theta}{16\sec^4\theta\sqrt{4\sec^2\theta-4}} \\
&= \frac{1}{16}\int \frac{2\sec\theta\tan\theta\,d\theta}{\sec^4\theta\,2\tan\theta} \\
&= \frac{1}{16}\int \cos^3\theta\,d\theta \\
\text{(from Formula 17)} \quad &= \frac{1}{16}\left(\frac{1}{3}\sin\theta\cos^2\theta + \frac{2}{3}\int \cos\theta\,d\theta\right) \\
&= \frac{1}{48}\sin\theta\cos^2\theta + \frac{1}{24}\sin\theta + C
\end{aligned}
$$



From the triangle,

$$\cos\theta = \frac{2}{x}$$

and

$$\sin\theta = \frac{\sqrt{x^2-4}}{x}.$$

Thus

$$
\begin{aligned}
\int \frac{dx}{x^4\sqrt{x^2-4}} &= \frac{1}{48}\frac{4}{x^2}\frac{\sqrt{x^2-4}}{x} + \frac{1}{24}\frac{\sqrt{x^2-4}}{x} + C \\
&= \frac{1}{24}\frac{2\sqrt{x^2-4}}{x^3} + \frac{1}{24}\frac{\sqrt{x^2-4}}{x} + C \\
\text{or} \quad & \frac{(2+x^2)\sqrt{x^2-4}}{24x^3} + C
\end{aligned}
$$

Without Formula 17:

$$
\begin{aligned}
\int \cos^3\theta\,d\theta &= \int (1-\sin^2\theta)\cos\theta, d\theta \\
(u = \sin\theta) \quad &= \int (1-u^2)\,du \\
&= u - \frac{u^3}{3} + C \\
&= \sin\theta - \frac{\sin^3\theta}{3} + C
\end{aligned}
$$

4. [9 marks] Find $\displaystyle\int \frac{x^3+x^2+x+4}{x^4+5x^2+4}\,dx$

Solution: use the method of partial fractions. First factor the denominator:

$$x^4+5x^2+4=(x^2+4)(x^2+1)$$

and then let

$$\frac{x^3+x^2+x+4}{x^4+5x^2+4}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+1}.$$

Hence

$$\begin{aligned}\frac{x^3+x^2+x+4}{x^4+5x^2+4} &= \frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+1}.\\ &= \frac{(Ax+B)(x^2+1)+(Cx+D)(x^2+4)}{(x^2+4)(x^2+1)}\\ &= \frac{(A+C)x^3+(B+D)x^2+(A+4C)x+B+4D}{(x^2+4)(x^2+1)}\end{aligned}$$

Comparing numerators results in the following system of equations

$$\begin{cases} A + C = 1 \\ B + D = 1 \\ A + 4C = 1 \\ B + 4D = 4 \end{cases}$$

which has solution

$$(A,B,C,D)=(1,0,0,1).$$

So

$$\begin{aligned}\int \frac{x^3+x^2+x+4}{x^4+5x^2+4}\,dx &= \int\left(\frac{x}{x^2+4}+\frac{1}{x^2+1}\right)dx\\ &= \frac{1}{2}\ln(x^2+4)+\tan^{-1}x+K\end{aligned}$$

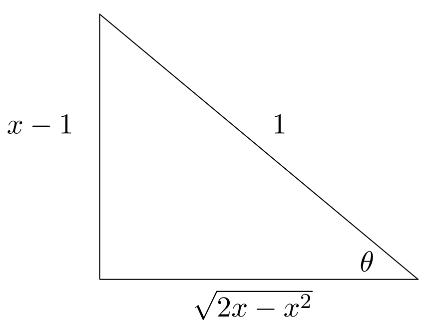
5. [9 marks] Find $\displaystyle\int \frac{x^2}{\sqrt{2x - x^2}}\, dx$

Solution: complete the square and use a trigonometric substitution.

$$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1 - 1) = 1 - (x-1)^2$$

Let $x - 1 = \sin\theta$, then $x = 1 + \sin\theta$ and

$$\begin{aligned}
\int \frac{x^2}{\sqrt{2x - x^2}}\, dx &= \int \frac{x^2}{\sqrt{1 - (x-1)^2}}\, dx \\
&= \int \frac{(1 + \sin\theta)^2}{\sqrt{1 - \sin^2\theta}} \cos\theta \, d\theta \\
&= \int (1 + 2\sin\theta + \sin^2\theta)\, d\theta, \text{ assuming } \sqrt{\cos^2\theta} = \cos\theta \\
&= \theta - 2\cos\theta + \frac{1}{2}\int (1 - \cos(2\theta))\, d\theta \\
&= \theta - 2\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin(2\theta) + C \\
&= \frac{3\theta}{2} - 2\cos\theta - \frac{\sin\theta\cos\theta}{2} + C
\end{aligned}$$



From the triangle,

$$\cos\theta = \sqrt{2x - x^2};$$

and

$$\sin\theta = x - 1$$

so

$$\begin{aligned}
\int \frac{x^2}{\sqrt{2x - x^2}}\, dx &= \frac{3}{2}\sin^{-1}(x-1) - 2\sqrt{2x - x^2} - \frac{(x-1)\sqrt{2x - x^2}}{2} + C \\
\text{or } & \frac{3}{2}\sin^{-1}(x-1) - \frac{1}{2}x\sqrt{2x - x^2} - \frac{3}{2}\sqrt{2x - x^2} + C
\end{aligned}$$

6. [9 marks] Determine if each of the following improper integrals converges or diverges. If it converges, find its value.

(a) [4 marks] $\int_0^\infty \left(e^{-x} + e^{-3x}\right)\, dx$

Solution: use the correct definition for this type of improper integral.

$$\begin{aligned}
\int_0^\infty \left(e^{-x} + e^{-3x}\right) dx &= \lim_{b\to\infty} \int_0^b \left(e^{-x} + e^{-3x}\right) dx \\
&= \lim_{b\to\infty} \left[-e^{-x} - \frac{e^{-3x}}{3}\right]_0^b \\
&= \lim_{b\to\infty} \left(-\frac{1}{e^b} - \frac{1}{3e^{3b}}\right) + 1 + \frac{1}{3} \\
&= 0 + \frac{4}{3} \\
&= \frac{4}{3}
\end{aligned}$$

(b) [5 marks] $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

Solution: let $x = t^2$. Then $dx = 2t\,dt$ and

$$\begin{aligned}
\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} &= \int_0^\infty \frac{1}{t}\frac{1}{t^2+1}\, 2t\, dt \\
&= 2\int_0^\infty \frac{dt}{t^2+1} \\
&= 2\lim_{b\to\infty} \left[\tan^{-1} t\right]_0^b \\
&= 2\lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0 \\
&= 2\left(\frac{\pi}{2}\right) - 0 \\
&= \pi
\end{aligned}$$

Alternate Solution: let $u = \tan^{-1}\sqrt{x}$. Then $du = \frac{1}{x+1}\frac{1}{2\sqrt{x}}\, dx$ and

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = 2\int_0^{\pi/2} du = \pi.$$

7. [7 marks] Find $\int 15\,x^{44}\sqrt{x^{15}+1}\,dx$

Solution: let $x^{15}+1=u^2$. Then $15x^{14}\,dx = 2u\,du$ and

$$\begin{aligned}
\int 15\,x^{44}\sqrt{x^{15}+1}\,dx &= \int (x^{15})^2\sqrt{x^{15}+1}(15\,x^{14})\,dx \\
&= \int (u^2-1)^2\,u\,2u\,du \\
&= 2\int u^2(u^4-2u^2+1)\,du \\
&= 2\int \left(u^6-2u^4+u^2\right)\,du \\
&= \frac{2}{7}u^7-\frac{4}{5}u^5+\frac{2}{3}u^3+C \\
&= \frac{2}{7}(x^{15}+1)^{7/2}-\frac{4}{5}(x^{15}+1)^{5/2}+\frac{2}{3}(x^{15}+1)^{3/2}+C
\end{aligned}$$

This expression can be factored, but we wouldn't expect anybody to do it:

$$\frac{2\sqrt{x^{15}+1}}{105}(x+1)(1-x+x^2-x^3+x^4)(1-x+x^2)(1+x-x^3-x^4-x^5+x^7+x^8)(15x^{30}-12x^{15}+8)$$