University of Toronto

Solutions to MAT 187H1S TERM TEST

of **Thursday, February 4, 2010**

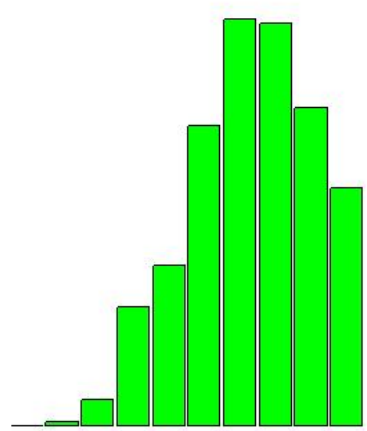
Duration: 90 minutes

General Comments:

1. In Questions 2 and 7 it really helps to make a suitable substitution before you use integration by parts.
2. Questions 2 and 3 are based on WileyPlus assigned homework problems.
3. Indefinite integral require a constant of integration; definite integrals do not.
4. Questions 1 and 3 require you to make a substituion into a definite integral. The correct way to do this is to change the limits of integration as you change the variable. Theorem 5.9.1 on page 391 of the textbook spells it out:
$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du,$$
where the substitution is $u = g(x)$. Doing this will actually save you work.
5. In Question 5, $6x - x^2 \geq 0 \Rightarrow -3 \leq x - 3 \leq 3$, so letting $x - 3 = 3\sec\theta$ makes no sense at all.
6. In both Question 6(b) and 7 the substitution $x = \sec\theta$ could be used to first simplify the integrand.
7. Any student who let $dv = e^{x^4}\, dx$, in Question 2, and then said $v = \dfrac{e^{x^4}}{4x^3}$, should think of retiring from calculus.

Breakdown of Results: 466 students wrote this test. The marks ranged from 16.7% to 100%, and the average was 67.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	12.0%
A	28.1%	80-89%	16.1%
B	20.4%	70-79%	20.4%
C	20.6%	60-69%	20.6%
D	15.2%	50-59%	15.2%
F	15.7%	40-49%	8.2%
		30-39%	6.0%
		20-29%	1.3%
		10-19%	0.2%
		0-9%	0.0%



1. [7 marks] Find $\displaystyle\int_0^{\pi/4} \sin^4 x \, \cos^3 x \, dx$

Solution: let $u = \sin x$. Then $du = \cos x \, dx$, and

$$
\begin{aligned}
\int_0^{\pi/4} \sin^4 x \, \cos^3 x \, dx &= \int_0^{\pi/4} \sin^4 x \, \cos^2 x \, \cos x \, dx \\
&= \int_0^{\pi/4} \sin^4 x \, (1 - \sin^2 x) \, \cos x \, dx \\
\text{(change limits!)} \quad &= \int_0^{1/\sqrt{2}} u^4 \left(1 - u^2\right) \, du \\
&= \int_0^{1/\sqrt{2}} \left(u^4 - u^6\right) \, du \\
&= \left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right]_0^{1/\sqrt{2}} \\
&= \frac{1}{20\sqrt{2}} - \frac{1}{56\sqrt{2}} \\
&= \frac{9}{280\sqrt{2}} \text{ or } \frac{9\sqrt{2}}{560}
\end{aligned}
$$

Alternate Solution: If you don't change the limits of integration as you substitute you must first find the indefinite integral. With the same choice of u as above:

$$\int \sin^4 x \, \cos^3 x \, dx = \int u^4 \left(1 - u^2\right) \, du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

So

$$\int \sin^4 x \, \cos^3 x \, dx = \left[\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right]_0^{\pi/4} = \frac{9\sqrt{2}}{560}.$$

2. [9 marks] Find $\int x^{11} e^{x^4}\, dx$

Solution: first let $y = x^4$, then use parts twice.

$$\begin{aligned}
\int x^{11} e^{x^4}\, dx &= \frac{1}{4}\int \left(x^4\right)^2 \cdot 4x^3 \cdot e^{x^4}\, dx \\
&= \frac{1}{4}\int y^2 e^y\, dy \\
(u = y^2, dv = e^y\, dy) &= \frac{1}{4}\left(uv - \int v\, du\right) \\
&= \frac{y^2 e^y}{4} - \frac{1}{2}\int y\, e^y\, dy \\
(s = y, dt = e^y\, dy) &= \frac{y^2 e^y}{4} - \frac{1}{2}\left(st - \int t\, ds\right) \\
&= \frac{y^2 e^y}{4} - \frac{1}{2}\left(ye^y - \int e^y\, dy\right) \\
&= \frac{y^2 e^y}{4} - \frac{ye^y}{2} + \frac{e^y}{2} + C \\
&= \frac{x^8 e^{x^4}}{4} - \frac{x^4 e^{x^4}}{2} + \frac{e^{x^4}}{2} + C \\
(\text{optionally}) &= \frac{e^{x^4}}{4}\left(x^8 - 2x^4 + 2\right) + C
\end{aligned}$$

Alternate Solution: or simply use parts twice.

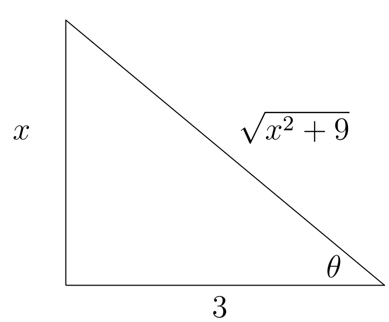
$$\begin{aligned}
\int x^{11} e^{x^4}\, dx &= \frac{1}{4}\int x^8 \cdot 4x^3 e^{x^4}\, dx \\
(u = x^8, dv = 4x^3 e^{x^4}\, dx) &= \frac{1}{4}\left(uv - \int v\, du\right) \\
&= \frac{x^8 e^{x^4}}{4} - \frac{1}{4}\int 8x^7\, e^{x^4}\, dx \\
&= \frac{x^8 e^{x^4}}{4} - \frac{1}{2}\int x^4\, 4x^3 e^{x^4}\, dx \\
(s = x^4, dt = 4x^3 e^{x^4}\, dx) &= \frac{x^8 e^{x^4}}{4} - \frac{1}{2}\left(st - \int t\, ds\right) \\
&= \frac{x^8 e^{x^4}}{4} - \frac{1}{2}\left(x^4 e^{x^4} - \int 4x^3 e^{x^4}\, dx\right) \\
&= \frac{x^8 e^{x^4}}{4} - \frac{x^4 e^{x^4}}{2} + \frac{e^{x^4}}{2} + C
\end{aligned}$$

3. [9 marks] Find $\displaystyle\int_{3}^{3\sqrt{3}} \frac{dx}{x^4\sqrt{x^2+9}}$

Solution: let $x = 3\tan\theta$; then $dx = 3\sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int_{3}^{3\sqrt{3}} \frac{dx}{x^4\sqrt{x^2+9}} &= \int_{\pi/4}^{\pi/3} \frac{3\sec^2\theta\, d\theta}{81\tan^4\theta\sqrt{9\tan^2\theta+9}} \\
&= \frac{1}{81}\int_{\pi/4}^{\pi/3} \frac{\sec\theta\, d\theta}{\tan^4\theta} \\
&= \frac{1}{81}\int_{\pi/4}^{\pi/3} \frac{\cos^3\theta}{\sin^4\theta}\, d\theta \\
&= \frac{1}{81}\int_{\pi/4}^{\pi/3} \frac{1-\sin^2\theta}{\sin^4\theta}\, \cos\theta\, d\theta \\
(u = \sin\theta) \quad &= \frac{1}{81}\int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{1-u^2}{u^4}\, du \\
&= \frac{1}{81}\int_{1/\sqrt{2}}^{\sqrt{3}/2} \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du \\
&= \frac{1}{81}\left[\frac{-1}{3u^3} + \frac{1}{u}\right]_{1/\sqrt{2}}^{\sqrt{3}/2} \\
&= \frac{1}{81}\left(\frac{-8}{9\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{2\sqrt{2}}{3} - \sqrt{2}\right) \\
&= \frac{10}{729\sqrt{3}} - \frac{\sqrt{2}}{243} \text{ or } \frac{10\sqrt{3}}{2187} - \frac{\sqrt{2}}{243}
\end{aligned}$$

Alternately: if you don't change limits of integration as you substitute you must find the antiderivative first. From the triangle, $u = \sin\theta = \dfrac{x}{\sqrt{x^2+9}}$ and so



$$\begin{aligned}
&\int \frac{dx}{x^4\sqrt{x^2+9}} \\
&= \frac{1}{81}\left(\frac{-1}{3u^3} + \frac{1}{u}\right) + C \\
&= \frac{\sqrt{x^2+9}}{81x} - \frac{(x^2+9)^{3/2}}{243x^3} + C
\end{aligned}$$

4. [9 marks] Find $\displaystyle\int \frac{4x^3-6x^2+6x}{(x-1)^2(x^2+1)}\,dx$

Solution: use the method of partial fractions. Let

$$\frac{4x^3-6x^2+6x}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Then

$$\begin{aligned}\frac{4x^3-6x^2+6x}{(x-1)^2(x^2+1)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} \\ &= \frac{A(x^2+1)(x-1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}\end{aligned}$$

Comparing numerators results in the equation

$$4x^3-6x^2+6x = (A+C)x^3 + (B-A-2C+D)x^2 + (A+C-2D)x - A + B + D.$$

Then comparing coefficients results in the following system of equations

$$\begin{cases} A + C = 4 \\ -A + B - 2C + D = -6 \\ A + C - 2D = 6 \\ -A + B + D = 0 \end{cases}$$

which has solution

$$(A,B,C,D) = (1,2,3,-1).$$

So

$$\begin{aligned}\int \frac{4x^3-6x^2+6x}{(x-1)^2(x^2+1)}\,dx &= \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}\right) dx \\ &= \int \frac{dx}{x-1} + \int \frac{2dx}{(x-1)^2} + \int \frac{3x\,dx}{x^2+1} - \int \frac{dx}{x^2+1} \\ &= \ln|x-1| - \frac{2}{x-1} + \frac{3}{2}\ln(x^2+1) - \tan^{-1}x + K\end{aligned}$$

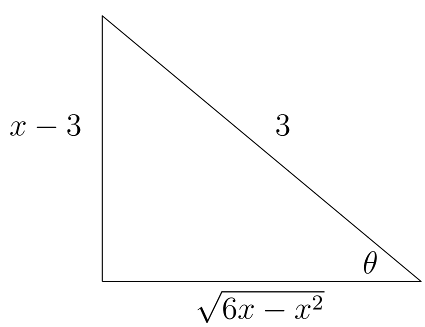
5. [9 marks] Find $\int \frac{x+5}{\sqrt{6x-x^2}}\,dx$

Solution: complete the square and use a trigonometric substitution.

$$6x - x^2 = -(x^2 - 6x) = -(x^2 - 6x + 9 - 9) = 9 - (x-3)^2$$

Let $x - 3 = 3\sin\theta$, then $x = 3 + 3\sin\theta$ and

$$\begin{aligned}
\int \frac{x+5}{\sqrt{6x-x^2}}\,dx &= \int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx \\
&= \int \frac{3+3\sin\theta+5}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta \\
&= \int (3\sin\theta + 8)\,d\theta, \text{ assuming } \sqrt{\cos^2\theta} = \cos\theta \\
&= -3\cos\theta + 8\,\theta + C
\end{aligned}$$



From the triangle, or using the basic trig identity:

$$\cos\theta = \frac{\sqrt{6x-x^2}}{3}.$$

So

$$\int \frac{x+5}{\sqrt{6x-x^2}}\,dx = -\sqrt{6x-x^2} + 8\sin^{-1}\left(\frac{x-3}{3}\right) + C$$

6. [9 marks] Determine if each of the following improper integrals converges or diverges. If it converges, find its value.

(a) [3 marks] $\int_0^1 \frac{1}{x^3}\,dx$

Solution: use the correct definition for this type of improper integral.

$$\begin{aligned} \int_0^1 \frac{1}{x^3}\,dx &= \lim_{a\to 0^+} \int_a^1 \frac{1}{x^3}\,dx \\ &= \lim_{a\to 0^+} \left[-\frac{1}{2x^2}\right]_a^1 \\ &= -\frac{1}{2} + \lim_{a\to 0^+} \frac{1}{2a^2} \\ &= \infty \end{aligned}$$

So this integral diverges.

(b) [6 marks] $\int_2^\infty \frac{dx}{x^3 - x}$

Solution: the substitution $x = \sec\theta$ changes this improper integral to a proper integral:

$$\begin{aligned} \int_2^\infty \frac{dx}{x^3 - x} &= \int_2^\infty \frac{dx}{x(x^2-1)} \\ &= \int_{\pi/3}^{\pi/2} \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\,(\sec^2\theta - 1)} \\ &= \int_{\pi/3}^{\pi/2} \cot\theta\,d\theta \\ &= [\ln\sin\theta]_{\pi/3}^{\pi/2} \\ &= \ln 1 - \ln\left(\frac{\sqrt{3}}{2}\right) \\ &= \ln\left(\frac{2}{\sqrt{3}}\right) \text{ or } \ln 2 - \frac{\ln 3}{2} \text{ or } \ln 2 - \ln\sqrt{3} \end{aligned}$$

So this integral converges.

Outline of an Alternate Solution: use partial fractions and the appropriate definition of improper integral.

$$\int_2^\infty \frac{dx}{x^3 - x} = \lim_{b\to\infty} \int_2^b \left(\frac{1}{2}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{x}\right) dx = \lim_{b\to\infty} \left[\ln\frac{\sqrt{x^2-1}}{x}\right]_2^b = -\ln\frac{\sqrt{3}}{2}$$

7. [8 marks] Find $\displaystyle\int x \sec^{-1}\sqrt{x^2-1}\,dx$, for $x > \sqrt{2}$.

Solution: let $t = \sqrt{x^2-1}$. Then $t > 1$, $t^2 = x^2 - 1$ and $2t\,dt = 2x\,dx \Leftrightarrow t\,dt = x\,dx$. After the change of variables, use integration by parts.

$$\begin{aligned}
\int x \sec^{-1}\sqrt{x^2-1}\,dx &= \int t\sec^{-1} t\,dt \\
(u = \sec^{-1} t, dv = t\,dt) &= uv - \int v\,du \\
&= \frac{t^2}{2}\sec^{-1} t - \frac{1}{2}\int \frac{t^2}{t\sqrt{t^2-1}}\,dt \\
&= \frac{t^2}{2}\sec^{-1} t - \frac{1}{2}\int \frac{t}{\sqrt{t^2-1}}\,dt \\
&= \frac{t^2}{2}\sec^{-1} t - \frac{1}{2}\sqrt{t^2-1} + C \\
&= \frac{(x^2-1)}{2}\sec^{-1}\sqrt{x^2-1} - \frac{1}{2}\sqrt{x^2-2} + C
\end{aligned}$$

Alternate Solutions: $u = \sec^{-1}\sqrt{x^2-1}$, $dv = x\,dx$, with a clever choice for v :

$$\begin{aligned}
&\int x \sec^{-1}\sqrt{x^2-1}\,dx = uv - \int v\,du \\
&= \frac{(x^2-1)}{2}\sec^{-1}\sqrt{x^2-1} - \int \frac{x^2-1}{2}\frac{1}{\sqrt{x^2-1}\sqrt{x^2-1-1}}\frac{x}{\sqrt{x^2-1}}\,dx \\
&= \frac{(x^2-1)}{2}\sec^{-1}\sqrt{x^2-1} - \frac{1}{2}\int \frac{x}{\sqrt{x^2-2}}\,dx \\
&= \frac{(x^2-1)}{2}\sec^{-1}\sqrt{x^2-1} - \frac{1}{2}\sqrt{x^2-2} + C
\end{aligned}$$

If you just let $v = \dfrac{x^2}{2}$, then you end up with the integral $\displaystyle\int \frac{x^3\,dx}{(x^2-1)\sqrt{x^2-2}}$, which can be handled by substituting $w^2 = x^2 - 2$, resulting in

$$\int \frac{x^3\,dx}{(x^2-1)\sqrt{x^2-2}} = \int \frac{(w^2+2)w\,dw}{(w^2+1)w} = \int \left(1 + \frac{1}{w^2+1}\right)dw;$$

or by writing

$$\int \frac{x^3\,dx}{(x^2-1)\sqrt{x^2-2}} = \int \frac{(x^3-x+x)\,dx}{(x^2-1)\sqrt{x^2-2}} = \int \frac{x\,dx}{\sqrt{x^2-2}} + \int \frac{x\,dx}{(x^2-1)\sqrt{x^2-2}}$$

and now letting $w^2 = x^2 - 2$ in this last integral.