University of Toronto

Solutions to MAT 187H1S TERM TEST

Thursday, February 5, 2009

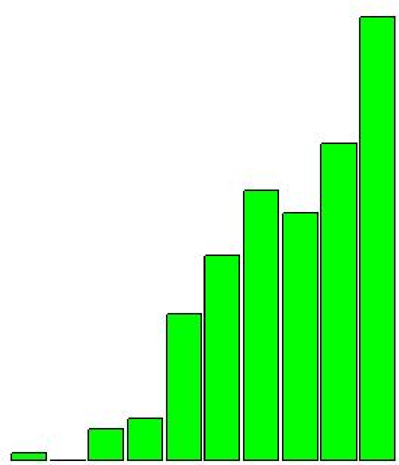
Duration: 90 minutes

General Comments: More than a quarter of all students who wrote this test got A^+. This was not surprising, since all the questions could be done by routine methods.

1. The most frequent problem for students who did not do well, aside from picking the wrong method, was incorrect algebraic manipulation.
2. There are still many students who use the $=$ and $\Rightarrow$ symbols incorrectly. This will cost you marks.
3. An indefinite integral requires a constant of integration, but a definite integral doesn't.
4. Questions 2 and 4 require you to make a substituion into a definite integral. The correct way to do this is to change the limits of integration as you change the variable. Theorem 1 on page 377 of the textbook spells it out:
$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du,$$
where the substitution is $u = g(x)$. Doing this will actually save you work.
5. In Question 5, $2x - x^2 \geq 0 \Rightarrow -1 \leq x - 1 \leq 1$, so letting $x - 1 = \sec\theta$ makes no sense at all.

Breakdown of Results: 443 students wrote this test. The marks ranged from 6.7% to 100%, and the average was 72.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	26.0%
A	44.5%	80-89%	18.5%
B	14.4%	70-79%	14.4 %
C	15.8%	60-69%	15.8%
D	12.0%	50-59%	12.0%
F	13.3%	40-49%	8.6%
		30-39%	2.5%
		20-29%	1.8%
		10-19%	0.0%
		0-9%	0.4%



1. [8 marks] Determine if each of the following improper integrals converges or diverges. If it converges, find its value.

   (a) [3 marks] $\int_0^1 \frac{1}{x^2}\,dx$

   Solution: use the correct definition for this type of improper integral.

$$\begin{aligned}\int_0^1 \frac{1}{x^2}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\,dx \\ &= \lim_{a\to 0^+}\left[-\frac{1}{x}\right]_a^1 \\ &= -1+\lim_{a\to 0^+}\frac{1}{a} \\ &= \infty\end{aligned}$$

   So this integral diverges.

   (b) [5 marks] $\int_0^\infty xe^{-x}\,dx$

   Solution: use the correct definition for this type of improper integral.

$$\begin{aligned}\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\,dx \\ &= \lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b,\ \text{by parts} \\ &= \lim_{b\to\infty}\left(-\frac{b}{e^b}-\frac{1}{e^b}\right)+0+1 \\ &= \lim_{b\to\infty}\left(-\frac{b}{e^b}\right)-\lim_{b\to\infty}\left(\frac{1}{e^b}\right)+1 \\ &= \lim_{b\to\infty}\left(-\frac{1}{e^b}\right)-0+1,\ \text{by L'Hopital's rule} \\ &= 0+1=1\end{aligned}$$

   So this integral converges, and its value is 1.

   Alternate Solution: By the definition of the Gamma function,

$$\int_0^\infty xe^{-x}\,dx=\Gamma(2)=1!=1.$$

2. [8 marks] Find $\displaystyle\int_0^{\pi/4} \tan^5 x \sec^4 x \, dx$

Solution: let $u = \tan x$. Then $du = \sec^2 x \, dx$, and

$$\begin{aligned}
\int_0^{\pi/4} \tan^5 x \sec^4 x \, dx &= \int_0^{\pi/4} \tan^5 x \sec^2 x \sec^2 x \, dx \\
&= \int_0^{\pi/4} \tan^5 x \, (\tan^2 x + 1) \sec^2 x \, dx \\
&= \int_0^{1} u^5 \left(u^2 + 1\right) du \\
&= \int_0^{1} \left(u^7 + u^5\right) du \\
&= \left[\frac{1}{8}u^8 + \frac{1}{6}u^6\right]_0^1 \\
&= \frac{1}{8} + \frac{1}{6} \\
&= \frac{7}{24}
\end{aligned}$$

Alternate Solution: If you don't change the limits of integration as you substitute you must first find the indefinite integral. With the same choice of u as above:

$$\int \tan^5 x \sec^4 x \, dx = \int \left(u^7 + u^5\right) du = \frac{1}{8}u^8 + \frac{1}{6}u^6 + C = \frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 x + C.$$

So

$$\int_0^{\pi/4} \tan^5 x \sec^4 x \, dx = \left[\frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 x\right]_0^{\pi/4} = \frac{1}{8} + \frac{1}{6} = \frac{7}{24}.$$

3. [9 marks] Find $\int \frac{x^2+x}{x^4-1}\,dx$

Solution: $x^4-1=(x^2+1)(x+1)(x-1)$, so let

$$\frac{x^2+x}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}.$$

Then

$$\begin{aligned}\frac{x^2+x}{x^4-1} &= \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\\ &= \frac{A(x^2+1)(x+1)+B(x^2+1)(x-1)+(Cx+D)(x^2-1)}{x^4-1}\\ \Leftrightarrow x^2+x &= (A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+A-B-D\end{aligned}$$

Solve

$$\left\{\begin{array}{ccccccccc} A & + & B & + & C & & & = & 0\\ A & - & B & & & + & D & = & 1\\ A & + & B & - & C & & & = & 1\\ A & - & B & & & - & D & = & 0\end{array}\right. \Leftrightarrow (A,B,C,D)=(\frac{1}{2},0,-\frac{1}{2},\frac{1}{2}).$$

So

$$\begin{aligned}\int \frac{x^2+x}{x^4-1}\,dx &= \int\left(\frac{1}{2}\frac{1}{x-1}+\frac{1}{2}\frac{1-x}{x^2+1}\right)dx\\ &= \frac{1}{2}\int\frac{1}{x-1}\,dx+\frac{1}{2}\int\frac{1}{x^2+1}\,dx-\frac{1}{2}\int\frac{x}{x^2+1}\,dx\\ &= \frac{1}{2}\ln|x-1|+\frac{1}{2}\arctan x-\frac{1}{4}\ln(x^2+1)+K\end{aligned}$$

Alternately: you can simplify your algebra a bit by observing that

$$\frac{x^2+x}{x^4-1}=\frac{x(x+1)}{(x^2+1)(x+1)(x-1)}=\frac{x}{(x^2+1)(x-1)}.$$

Now find the partial fraction decomposition of

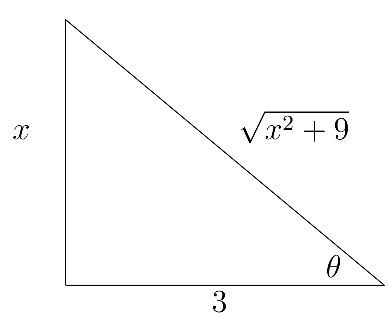
$$\frac{x}{(x^2+1)(x-1)}=\frac{A}{x-1}+\frac{Cx+D}{x^2+1}.$$

4. [9 marks] Find $\displaystyle\int_0^3 \frac{1}{(x^2+9)^2}\,dx$

Solution: let $x = 3\tan\theta$; then $dx = 3\sec^2\theta\,d\theta$ and

$$\begin{aligned}
\int_0^3 \frac{1}{(x^2+9)^2}\,dx &= \int_0^{\pi/4} \frac{1}{(9\tan^2\theta+9)^2}\,3\sec^2\theta\,d\theta \\
&= \int_0^{\pi/4} \frac{3\sec^2\theta}{(9\sec^2\theta)^2}\,d\theta \\
&= \frac{1}{27}\int_0^{\pi/4} \cos^2\theta\,d\theta \\
&= \frac{1}{27}\int_0^{\pi/4} \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{1}{54}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} \\
&= \frac{1}{54}\left(\frac{\pi}{4}+\frac{1}{2}\right) \\
&= \frac{\pi}{216}+\frac{1}{108}
\end{aligned}$$

Alternately: if you don't change limits of integration as you substitute you must find the antiderivative first. Which means you have to use another double angle formula and a triangle.



$$\begin{aligned}
\frac{1}{2}\sin(2\theta) &= \sin\theta\cos\theta \\
&= \frac{x}{\sqrt{x^2+9}}\frac{3}{\sqrt{x^2+9}} \\
&= \frac{3x}{x^2+9}
\end{aligned}$$

So

$$\int_0^3 \frac{1}{(x^2+9)^2}\,dx = \frac{1}{54}\left[\arctan\left(\frac{x}{3}\right)+\frac{3x}{x^2+9}\right]_0^3 = \frac{1}{54}\left(\frac{\pi}{4}+\frac{1}{2}\right).$$

NOTE: $\arctan 1 = \pi/4$, not $45°$.

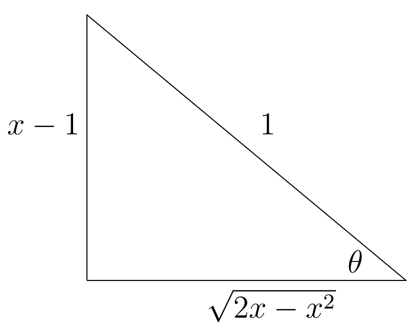
5. [9 marks] Find $\int (x+5)\sqrt{2x-x^2}\,dx$

Solution: complete the square and use a trigonometric substitution.

$$2x - x^2 = -(x^2-2x) = -(x^2-2x+1-1) = 1-(x-1)^2$$

Let $x-1=\sin\theta$, then:

$$\begin{aligned}
\int (x+5)\sqrt{2x-x^2}\,dx &= \int (x+5)\sqrt{1-(x-1)^2}\,dx \\
&= \int (\sin\theta+1+5)\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \\
&= \int (\sin\theta+6)\cos^2\theta\,d\theta \\
&= \int \sin\theta\cos^2\theta\,d\theta + 6\int \cos^2\theta\,d\theta \\
&= -\frac{1}{3}\cos^3\theta + 6\int \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= -\frac{1}{3}\cos^3\theta + 3\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\
&= -\frac{1}{3}\cos^3\theta + 3\theta + 3\sin\theta\cos\theta + C
\end{aligned}$$



From the triangle, or using the basic trig identity:

$$\cos\theta = \sqrt{2x-x^2}$$

So

$$\int (x+5)\sqrt{2x-x^2}\,dx = -\frac{1}{3}(2x-x^2)^{3/2} + 3\arcsin(x-1) + 3(x-1)\sqrt{2x-x^2} + C$$

6. [9 marks] Find $\int \sqrt{x} \arctan \sqrt{x}\, dx$

Solution: use integration by parts.

$$\begin{aligned}
\int \sqrt{x} \arctan \sqrt{x}\, dx &= \int u\, dv, \text{ with } u = \arctan \sqrt{x}, dv = \sqrt{x}\, dx \\
&= uv - \int v\, du \\
&= \frac{2}{3} x^{3/2} \arctan \sqrt{x} - \int \frac{2}{3} x^{3/2} \frac{1}{1 + (\sqrt{x})^2} \frac{1}{2\sqrt{x}}\, dx \\
&= \frac{2}{3} x^{3/2} \arctan \sqrt{x} - \frac{1}{3} \int \frac{x}{1+x}\, dx \\
&= \frac{2}{3} x^{3/2} \arctan \sqrt{x} - \frac{1}{3} \int \left(1 - \frac{1}{1+x}\right) dx \\
&= \frac{2}{3} x^{3/2} \arctan \sqrt{x} - \frac{1}{3} \left(x - \ln|1+x|\right) + C \\
&= \frac{2}{3} x^{3/2} \arctan \sqrt{x} - \frac{x}{3} + \frac{1}{3} \ln(1+x) + C, \text{ since } x \geq 0
\end{aligned}$$

7. [8 marks] Find $\displaystyle\int \frac{1+7e^x}{\sqrt{e^{2x}-1}}\,dx$, if $x > 0$.

Solution: Use a trig substitution. Since $x > 0$ it follows that $e^{2x} > 1$. Therefore you must use

$$e^x = \sec\theta \Leftrightarrow x = \ln\sec\theta.$$

Then

$$\begin{aligned}
&\int \frac{1+7e^x}{\sqrt{e^{2x}-1}}\,dx \\
=\ &\int \frac{1+7\sec\theta}{\sqrt{\sec^2\theta-1}}\,\frac{\sec\theta\tan\theta}{\sec\theta}\,d\theta \\
=\ &\int \frac{1+7\sec\theta}{\tan\theta}\,\tan\theta\,d\theta \\
=\ &\int (1+7\sec\theta)\,d\theta \\
=\ &\theta + 7\ln|\sec\theta+\tan\theta| + C \\
=\ &\sec^{-1}(e^x) + 7\ln|e^x+\sqrt{e^{2x}-1}| + C, \text{ since } \tan\theta = \sqrt{\sec^2\theta-1} \\
=\ &\sec^{-1}(e^x) + 7\ln(e^x+\sqrt{e^{2x}-1}) + C, \text{ since } e^x > 0
\end{aligned}$$

Alternate Solution: Let $u = e^x$ first. Then $du = e^x\,dx$ and

$$\begin{aligned}
\int \frac{1+7e^x}{\sqrt{e^{2x}-1}}\,dx &= \int \frac{1+7e^x}{e^x\sqrt{e^{2x}-1}}\,e^x\,dx \\
&= \int \frac{1+7u}{u\sqrt{u^2-1}}\,du \\
&= \int \frac{1}{u\sqrt{u^2-1}}\,du + 7\int \frac{1}{\sqrt{u^2-1}}\,du \\
&= \sec^{-1}u + 7\int \frac{1}{\sqrt{u^2-1}}\,du
\end{aligned}$$

Now use the trig substitution $u = \sec\theta$ in the last integral and you will get the same answer as in the first method.