University of Toronto

Solutions to MAT 187H1S TERM TEST

Thursday, February 7, 2008

Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

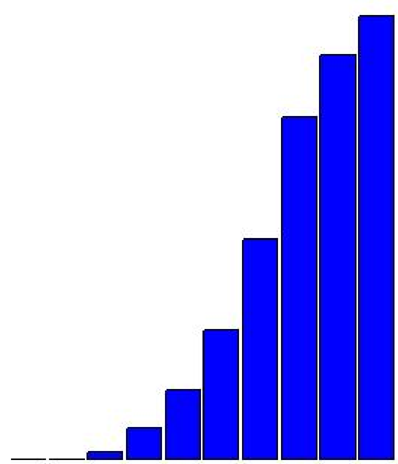
Instructions: Make sure this test has 8 pages. Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parantheses beside the question number.

TOTAL MARKS: 60

General Comments: this was a test consisting entirely of routine questions. It's nice to see that most of you took advantage of this and did very well. However, the one question that caused the most problem, Question 7, was right out of the homework: Section 7.3, #23. More of you should have aced that question!

Breakdown of Results: 472 students wrote this test. The marks ranged from 28% to 100%, and the average was 77.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	26.9%
A	51.5%	80-89%	24.6%
B	20.8%	70-79%	20.8%
C	13.3%	60-69%	13.3%
D	7.8%	50-59%	7.8%
F	6.6%	40-49%	4.2%
		30-39%	1.9%
		20-29%	0.4%
		10-19%	0.0%
		0-9%	0.0 %



1. [8 marks] Find $\displaystyle\int \frac{\sqrt{x}}{\left(x^{3/2}+5\right)^4}\,dx$

Solution: let $u = x^{3/2} + 5$. Then $du = \frac{3}{2}\sqrt{x}\,dx$, and

$$\begin{aligned}
\int \frac{\sqrt{x}}{\left(x^{3/2}+5\right)^4}\,dx &= \int \frac{1}{u^4}\,\frac{2}{3}\,du \\
&= \frac{2}{3}\int u^{-4}\,du \\
&= \frac{2}{3}\left(\frac{-1}{3}\right)u^{-3} + C \\
&= -\frac{2}{9}\frac{1}{\left(x^{3/2}+5\right)^3} + C
\end{aligned}$$

2. [8 marks] Find $\displaystyle\int \sin^4 x \, \cos^3 x \, dx$

Solution: let $u = \sin x$. Then $du = \cos x \, dx$, and

$$\begin{aligned}
\int \sin^4 x \, \cos^3 x \, dx &= \int \sin^4 x \cos^2 x \cos x \, dx \\
&= \int \sin^4 x \, (1 - \sin^2 x) \, \cos x \, dx \\
&= \int u^4 (1 - u^2) \, du \\
&= \int \left(u^4 - u^6 \right) \, du \\
&= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C \\
&= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C
\end{aligned}$$

3. [8 marks] Find $\displaystyle\int \frac{x^2+x}{(x^2+1)(x-1)}\,dx$

Solution: let

$$\frac{x^2+x}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Then

$$\begin{aligned} \frac{x^2+x}{(x^2+1)(x-1)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \\ &= \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)} \\ \Leftrightarrow x^2+x &= (A+B)x^2+(C-B)x+A-C \end{aligned}$$

Solve

$$\left\{\begin{array}{rrrrrrl} A & + & B & & & = & 1 \\ & - & B & + & C & = & 1 \\ A & & & - & C & = & 0 \end{array}\right. \Leftrightarrow (A,B,C) = (1,0,1).$$

So

$$\begin{aligned} \int \frac{x^2+x}{(x^2+1)(x-1)}\,dx &= \int \left(\frac{1}{x-1} + \frac{1}{x^2+1}\right)dx \\ &= \ln|x-1| + \tan^{-1}x + C \end{aligned}$$

Alternately: You can find the partial fraction decomposition by doing very little work:

$$\frac{x^2+x}{(x^2+1)(x-1)} = \frac{x^2+1+x-1}{(x^2+1)(x-1)} = \frac{1}{x-1} + \frac{1}{x^2+1}.$$

4. [8 marks] Find $\displaystyle\int_1^2 \frac{1}{\sqrt{(2-x)(x-1)}}\,dx$

Solution: complete the square and use a trigonometric substitution.

$$\begin{aligned}
(2-x)(x-1) &= -2+3x-x^2 \\
&= -(x^2-3x+2) \\
&= -\left(x^2-3x+\frac{9}{4}-\frac{1}{4}\right) \\
&= -\left(x-\frac{3}{2}\right)^2+\frac{1}{4} \\
&= \frac{1}{4}-\left(x-\frac{3}{2}\right)^2
\end{aligned}$$

Let

$$x-\frac{3}{2}=\frac{1}{2}\sin\theta;$$

then, putting everything in terms of θ :

$$\begin{aligned}
\int_1^2 \frac{1}{\sqrt{(2-x)(x-1)}}\,dx &= \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta}}\,\frac{1}{2}\cos\theta\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{2}{\sqrt{\cos^2\theta}}\,\frac{1}{2}\cos\theta\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} d\theta, \text{ since } \cos\theta\ge 0 \text{ for } \theta\in[-\pi/2,\pi/2] \\
&= [\theta]_{-\pi/2}^{\pi/2} \\
&= \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}$$

Notes:

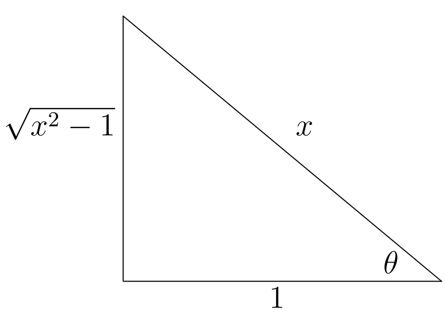
1. The integral is improper, but the limits of integration are no longer a problem once you have the antiderivative.
2. In terms of x,

$$\int_1^2 \frac{1}{\sqrt{(2-x)(x-1)}}\,dx = \left[\sin^{-1}(2x-3)\right]_1^2.$$

5. [10 marks] Find $\displaystyle\int \frac{\sqrt{x^2-1}}{x^3}\,dx$

Solution: let $x = \sec\theta$; then $dx = \sec\theta\tan\theta\,d\theta$ and

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x^3}\,dx &= \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^3\theta}\,\sec\theta\tan\theta\,d\theta \\
&= \int \frac{\tan\theta}{\sec^3\theta}\,\sec\theta\tan\theta\,d\theta \\
&= \int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\
&= \int \frac{\sec^2\theta - 1}{\sec^2\theta}\,d\theta \\
&= \int \left(1 - \cos^2\theta\right)\,d\theta \\
&= \int \left(1 - \frac{1+\cos(2\theta)}{2}\right)\,d\theta \\
&= \frac{1}{2}\int \left(1 - \cos(2\theta)\right)\,d\theta \\
&= \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{1}{2}\left(\theta - \sin\theta\cos\theta\right) + C
\end{aligned}$$



In the triangle, $\sec\theta = x$. Then

$$\sin\theta = \frac{\sqrt{x^2-1}}{x} \text{ and } \cos\theta = \frac{1}{x}.$$

So

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x^3}\,dx &= \frac{1}{2}\left(\theta - \sin\theta\cos\theta\right) + C \\
&= \frac{1}{2}\left(\sec^{-1}x - \frac{\sqrt{x^2-1}}{x}\frac{1}{x}\right) + C \\
&= \frac{1}{2}\left(\sec^{-1}x - \frac{\sqrt{x^2-1}}{x^2}\right) + C \\
&= \frac{1}{2}\sec^{-1}x - \frac{\sqrt{x^2-1}}{2x^2} + C
\end{aligned}$$

6. [10 marks] Find $\int_0^\infty e^{-x} \sin x \, dx$

Solution: use integration by parts to find the antiderivative; then worry about the improper limit.

$$
\begin{aligned}
\int e^{-x} \sin x \, dx &= \int u \, dv, \text{ with } u = e^{-x}, dv = \sin x \, dx \\
&= e^{-x}(-\cos x) - \int e^{x}(-\cos x)\,(-dx) \\
&= -e^{-x} \cos x - \int e^{-x} \cos x \, dx \\
&= -e^{-x} \cos x - \int s \, dt, \text{ with } s = e^{-x}, dt = \cos x \, dx \\
&= -e^{-x} \cos x - (st - \int t \, ds) \\
&= -e^{-x} \cos x - e^{-x} \sin x + \int e^{-x} \sin x \, (-dx) \\
&= -e^{-x}(\cos x + \sin x) - \int e^{-x} \sin x \, dx \\
\Rightarrow 2 \int e^{-x} \sin x \, dx &= -e^{-x}(\cos x + \sin x) + C \\
\Rightarrow \int e^{-x} \sin x \, dx &= -\frac{1}{2} e^{-x}(\cos x + \sin x) + C' \\
\Rightarrow \int_0^\infty e^{-x} \sin x \, dx &= \lim_{b \to \infty} \left[-\frac{1}{2} e^{-x}(\cos x + \sin x) \right]_0^b \\
&= -\frac{1}{2} \lim_{b \to \infty} \frac{\cos b + \sin b}{e^b} + \frac{1}{2} \\
&= 0 + \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

Note:

$$\lim_{b \to \infty} \frac{\cos b + \sin b}{e^b} = 0$$

by the squeeze law, not by L'Hopital's rule.

7. [8 marks] Find $\displaystyle\int \sec^{-1}\sqrt{x}\,dx$ (Recall: $\sec^{-1}$ is the same as arcsec.)

Solution: Use integration by parts, and be careful!

$$\begin{aligned}
\int \sec^{-1}\sqrt{x}\,dx = \int u\,dv &= uv - \int v\,du, \text{ with } u = \sec^{-1}\sqrt{x}, dv = dx \\
&= x\sec^{-1}\sqrt{x} - \int \frac{1}{\sqrt{x}\sqrt{x-1}}\,\frac{1}{2\sqrt{x}}\,x\,dx \\
&= x\sec^{-1}\sqrt{x} - \frac{1}{2}\int \frac{1}{\sqrt{x-1}}\,dx \\
&= x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C
\end{aligned}$$