

UNIVERSITY OF TORONTO, FACULTY OF APPLIED SCIENCE AND ENGINEERING

MAT187H1S - Calculus II

Final Exam - April 24, 2015

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Duration: 150 minutes.

No Aids permitted.

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First

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages and a detached **formula sheet**. Make sure you have all of them.
- You can use pages 13–14 for rough work or to complete a question (**Mark clearly**).

DO NOT DETACH PAGES 13–14.

GOOD LUCK!

**PART I.** No explanation is necessary.**(20 Marks)**For questions 1-4, consider the differential equation for  $y(t)$ :

$$y' + ty + t = 0.$$

1. The equilibrium solution is (circle one option)

(a)  $y(t) = -1$

(c)  $y(t) = 1$

(b)  $y(t) = 0$

(d) There is no equilibrium solution

2. If a solution
- $y(t)$
- has a horizontal asymptote, then what is it?

$$y = \underline{\hspace{10cm}}.$$

3. Consider the solution
- $y(t)$
- which satisfies
- $y(0) = 1$
- . Complete the blanks:

$$y'(0) = \underline{\hspace{10cm}},$$

$$y''(0) = \underline{\hspace{10cm}}.$$

4. (Harder!) Consider the solution
- $y(t)$
- which satisfies
- $y(0) = 1$
- . What is the first value
- $T > 0$
- for which
- $y(t) < 0$
- for all
- $t > T$
- ?

$$T = \underline{\hspace{10cm}}.$$

5. Consider two particles with positions
- $r_1(t) = (\cos(t), \frac{t}{2})$
- and
- $r_2(t) = (1 + t^2, \pi e^t)$
- .

(a) These particles collide and the paths intersect.

(b) The particles collide, but the paths don't intersect.

(c) The particles don't collide, but the paths intersect.

(d) The particles don't collide and the paths don't intersect.

Continued...



For questions 6–9, please consider the following power series

$$f(x) = \sum_{k=0}^{\infty} \frac{k+1}{4^k} x^k.$$

6. The radius of convergence for this series is  $R =$  \_\_\_\_\_.

7.  $\int f(x) dx =$  \_\_\_\_\_ (not as a power series).

8.  $f(x) =$  \_\_\_\_\_ (not as a power series).

9.  $\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)}{2^k} =$  \_\_\_\_\_.

10. The position of a particle is  $\vec{r}(t) = (x(t), y(t))$  as given in the figure on the right.

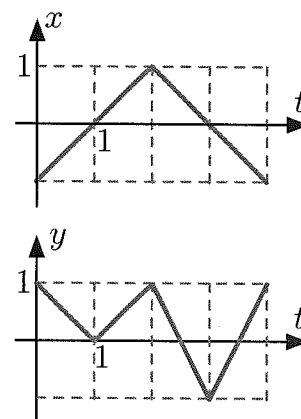
(a) The slope of the path of the particle at the point  $(\frac{1}{2}, 0)$  is

\_\_\_\_\_.

(b) At this point, the particle is moving to the

left / right and up / down .

(circle the correct options)





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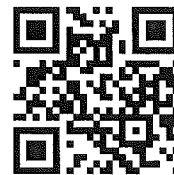
**PART II.** Answer the following questions. **Justify** your answers.

11. Consider the curve  $\vec{r}(t) = \left(-\frac{t^3}{3}, t^2, 1 - 2t\right)$ . **(20 Marks)**

(a) (10 marks) Find the maximum curvature.

(**Hint.** Use the formula with the cross product)

Continued...



(b) (6 marks) Show that there is no minimum curvature.

(c) (4 marks) Consider the particles with positions  $\vec{r}_1(t) = \vec{r}(t)$  and  $\vec{r}_2(t) = \vec{r}(e^3 - \pi t)$ . Will these particles collide? If so, when?

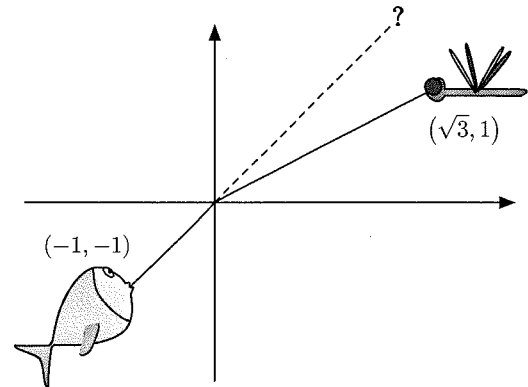


12. Archer fish hunt by spitting a jet of water at a nearby flying insect.

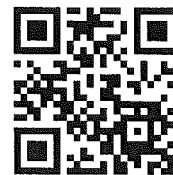
(20 Marks)

- (a) (5 marks) Assume that the fish's position, as shown, is  $(-1, -1)$  and the insect's position is  $(\sqrt{3}, 1)$  and the water surface is at  $y = 0$ .

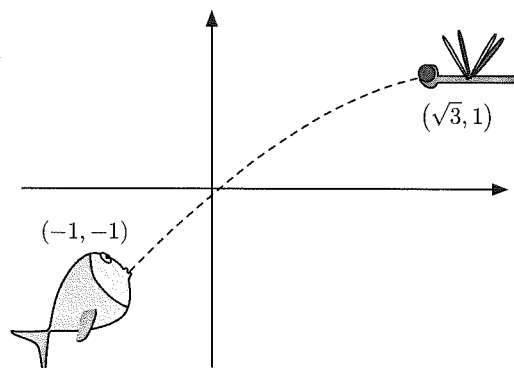
When the fish sees the insect, water refraction changes the angle of the light as in the figure on the right, causing the fish to misjudge the insect's position.



The fish sees the path to the insect as a straight line. What does the fish think is the position of the insect?



- (b) (10 marks) The fish hunts the insect by spitting a jet of water in the direction of the dotted line. Ignoring the effects of water and air resistance, but considering gravity,  $\vec{a} = (0, -g)$ , how fast should the fish spit the water to hit the insect? Assume all distances given are in cm and the insect is not moving.



- (c) (5 marks) How much time does the insect have to move out of the way?

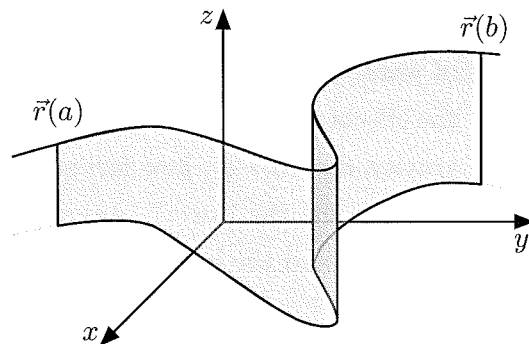


13. Consider a vector-valued function

$$\vec{r}(t) = (x(t), y(t), z(t)),$$

for  $t \in [a, b]$ . The graph of this function forms a curve.

Find a formula for the area between the curve and the  $xy$ -plane (as indicated in the figure).



(20 Marks)

**Hint.** To make the exercise easier, you can follow the steps:

(a) Let  $a < t_0 < t_1 < b$ . Sketch the rectangle with coordinates

$$P_0 = (x(t_0), y(t_0), 0) \quad , \quad P_1 = (x(t_0), y(t_0), z(t_1)) \quad , \quad P_2 = (x(t_1), y(t_1), z(t_1)) \quad , \quad P_3 = (x(t_1), y(t_1), 0)$$

(b) What is the area of the rectangle from (a)?

(c) Approximate the area (between the curve and the  $xy$ -plane) with the area of  $n$  thin rectangles.

(d) Find a formula for the exact area between the curve and the  $xy$ -plane by taking the limit as  $n \rightarrow \infty$ .





(Use this page to continue question 13)

**Bonus.** Test your formula. Use your formula to find the area for the function (4 Marks)

$\vec{r}(t) = (t, t, t)$  for  $t \in [0, 1]$  and confirm that it matches the formula for the area of a triangle.



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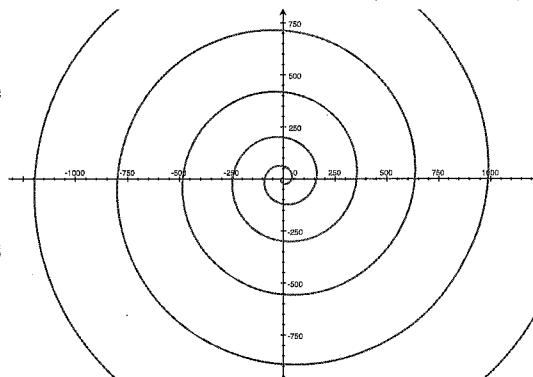
14. In this question we will investigate the shells of some land snails.

(20 Marks)

- (a) (8 marks) A certain snail's shell has the shape of the spiral

$$r = \theta^2,$$

where the magnitude of the angle  $\theta$  in radians equals the number of days since the snail was hatched.



If the variable  $r$  is in mm, how long will it take for the shell's spiral to be  $\frac{19}{3}$  mm long?

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- (b) (8 marks) A different snail's shell also forms a spiral. Each night, since it's humid, the length grows by  $\frac{1}{2n}$  cm, but during the day, since it's dry the length decreases by  $\frac{1}{2n+1}$  cm, where  $n$  is the number of days since it hatched.

If the snail lives forever, how long will the shell's spiral be?

(**Hint.** If you use  $\sum$  notation, it should remind you of a Taylor series, with  $x$  set equal to  $-1$ )



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- (c) (4 marks) For the same snail as in (b), how many days will it take for the shell's spiral to be within  $\frac{1}{100}$  cm of the limiting size?

**Bonus.** Excluding this bonus question, what is your mark on this exam? \_\_\_\_\_  $\pm 2$  points (**3 marks**)

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**USE THIS PAGE FOR ROUGH WORK.**

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**USE THIS PAGE FOR ROUGH WORK.**

The end.