University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2014**
First Year - CHE, CIV, IND, LME, MEC, MMS
MAT187H1S - CALCULUS II
Exam Type: A
Duration: 150 min.

General Comments:

1. Questions 1, 2, 3, 5, 6(a) and 8 were completely routine, and very similar to corresponding questions from last year's exam. They should have all been aced. Only Questions 6(b) and 7 could be classified as non-routine. Question 4 looks worse than it is; it certainly seemed to trip up lots of students. It was the only question with a failing average.

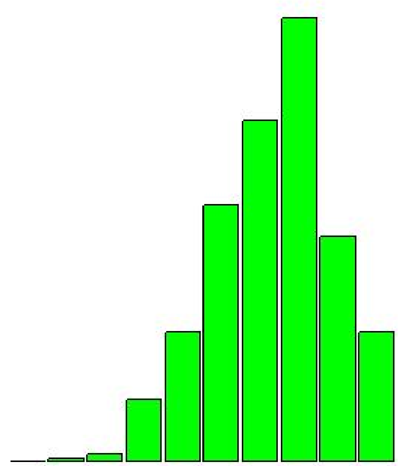
2. The integral in 1(b) can be done with almost any method you choose: parts, substitution, trig substitution, or even the substitution $u^2 = 1 + x^2$.

3. Question 4 is really straightforward: it comes down to integrating $1/(a^2+v^2)$ with respect to v, and $\tan u$ with respect to u. You just have to be careful. Many students thought they could use the method of the integrating factor, but it does not apply! Many students who separated variables correctly claimed the integral of $1/f(x)$ with respect to x is $\ln|f(x)|$, which is an egregious blunder!

4. In Questions 7 and 8 shockingly many students used nonsense like $\sqrt{a^2+b^2} = a + b$ or $(a+b)^2 = a^2+b^2$ to simplify their work. Many students forgot to use the chain rule in 8(a).

Breakdown of Results: 468 students wrote this exam. The marks ranged from 19% to 98%, and the average mark was 67.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.1%
A	22.2%	80-89%	14.1%
B	27.8%	70-79%	27.8%
C	21.4%	60-69%	21.4%
D	16.0%	50-59%	16.0%
F	12.6%	40-49%	8.1%
		30-39%	3.9%
		20-29%	0.4%
		10-19%	0.2%
		0-9%	0.0%



1. [avg: 10.1/13] Find the general solution for each of the following differential equations:

(a) [5 marks] $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y = 0$

Solution: solve the auxiliary quadratic equation:

$$r^2 + 2r + 10 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i.$$

Then the general solution is

$$y = Ae^{-x}\cos(3x) + B\,e^{-x}\sin(3x),$$

for arbitrary constants A and B.

(b) [8 marks] $\dfrac{dy}{dx} - \dfrac{x\,y}{1+x^2} = 4x^3$

Solution: use the method of the integrating factor.

$$\mu = e^{-\int (x/(1+x^2)\,dx} = e^{-(\ln(1+x^2))/2} = (1+x^2)^{-1/2} = \frac{1}{\sqrt{1+x^2}};$$

so

$$\begin{aligned}
y = \frac{1}{\mu}\int 4x^3\,\mu\,dx &= \sqrt{1+x^2}\int \frac{4x^3\,dx}{\sqrt{1+x^2}} \\
&= \sqrt{1+x^2}\int \frac{2x^2\,(2x\,dx)}{\sqrt{1+x^2}} \\
(\text{let } u = 1+x^2) \quad &= 2\sqrt{1+x^2}\int \frac{(u-1)\,du}{\sqrt{u}} \\
&= 2\sqrt{1+x^2}\int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du \\
&= 2\sqrt{1+x^2}\left(\frac{2}{3}u^{3/2} - 2\sqrt{u} + C\right) \\
&= 2\sqrt{1+x^2}\left(\frac{2}{3}(1+x^2)^{3/2} - 2\sqrt{1+x^2} + C\right) \\
&= \frac{4}{3}(1+x^2)^2 - 4(1+x^2) + 2C\sqrt{1+x^2} \\
&= \frac{4}{3}(1+x^2)(x^2-2) + K\sqrt{1+x^2},
\end{aligned}$$

for some arbitrary constant K.

2. [avg: 9.7/12] $x = e^{\sqrt{3}\,t}\cos t$ and $y = e^{\sqrt{3}\,t}\sin t$ are the parametric equations of a logarithmic spiral. Find the following:

(a) [6 marks] all points on the spiral, for $0 < t < \pi$, at which the tangent line to the curve is horizontal or vertical.

Solution:

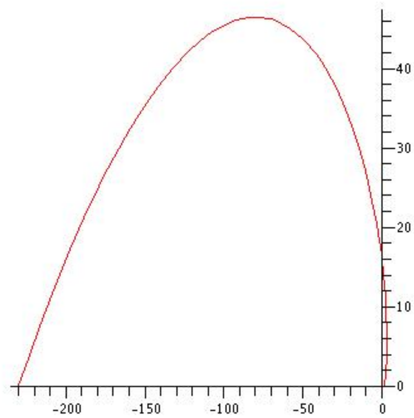
$$\frac{dx}{dt} = \sqrt{3}e^{\sqrt{3}t}\cos t - e^{\sqrt{3}t}\sin t; \quad \frac{dy}{dt} = \sqrt{3}e^{\sqrt{3}t}\sin t + e^{\sqrt{3}t}\cos t.$$

For horizontal tangents,

$$0 = \frac{dy}{dt} \Rightarrow \tan t = -\frac{1}{\sqrt{3}} \Rightarrow t = \frac{5\pi}{6},$$

and the point is

$$(x, y) = \left(-\frac{\sqrt{3}}{2}e^{5\pi\sqrt{3}/6}, \frac{1}{2}e^{5\pi\sqrt{3}/6}\right).$$



For vertical tangents, $0 = \dfrac{dx}{dt} \Rightarrow \tan t = \sqrt{3} \Rightarrow t = \dfrac{\pi}{3}$, and the point is

$$(x, y) = \left(\frac{1}{2}e^{\pi\sqrt{3}/3}, \frac{\sqrt{3}}{2}e^{\pi\sqrt{3}/3}\right).$$

(b) [6 marks] the length of the spiral, for $0 \le t \le \pi$.

Solution: let L be the length of the specified curve.

$$\begin{aligned}
L &= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^\pi \sqrt{e^{2\sqrt{3}t}(3\cos^2 t - 2\sqrt{3}\cos t\sin t + \sin^2 t + 3\sin^2 t + 2\sqrt{3}\sin t\cos t + \cos^2 t)}\,dt \\
&= \int_0^\pi \sqrt{4\,e^{2\sqrt{3}\,t}}\,dt = \int_0^\pi 2e^{\sqrt{3}\,t}\,dt \\
&= \left[\frac{2}{\sqrt{3}}e^{\sqrt{3}\,t}\right]_0^\pi = \frac{2}{\sqrt{3}}\left(e^{\pi\sqrt{3}} - 1\right)
\end{aligned}$$

Alternate Solution: the polar equation of this spiral is $r = e^{\sqrt{3}\,\theta}$, so its length is

$$L = \int_0^\pi \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \int_0^\pi \sqrt{3\,e^{2\sqrt{3}\,\theta} + e^{2\sqrt{3}\,\theta}}\,d\theta = \int_0^\pi 2e^{\sqrt{3}\,\theta}\,d\theta = \frac{2}{\sqrt{3}}\left(e^{\pi\sqrt{3}} - 1\right)$$

3. [avg: 13.4/15] Find all the critical points of the function $f(x, y) = x^2y + y^3 - 12y$ and determine if they are maximum points, minimum points, or saddle points.

Solution: Let $z = f(x, y)$. Then

$$\frac{\partial z}{\partial x} = 2xy \text{ and } \frac{\partial z}{\partial y} = x^2 + 3y^2 - 12.$$

Critical points:

$$\left.\begin{array}{rcl} x^2 + 3y^2 - 12 & = & 0 \\ 2xy & = & 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x^2 + 3y^2 & = & 12 \\ xy & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} x \text{ or } y = 0 \\ \text{and} \\ x^2 + 3y^2 = 12 \end{array}\right.$$

So the four critical points are:

$$(x, y) = (0, 2), (0, -2), (2\sqrt{3}, 0) \text{ or } (-2\sqrt{3}, 0).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 2y,\ \frac{\partial^2 z}{\partial y^2} = 6y,\ \frac{\partial^2 z}{\partial x \partial y} = 2x$$

and so

$$\Delta = (2y)(6y) - (2x)^2 = 12y^2 - 4x^2.$$

- At each of $(x, y) = (2\sqrt{3}, 0)$ or $(-2\sqrt{3}, 0)$

  $$\Delta = -48 < 0,$$

  so f has a saddle point at each of $(x, y, z) = (2\sqrt{3}, 0, 0)$ and $(-2\sqrt{3}, 0, 0)$
- At $(x, y) = (0, 2)$ $\Delta = 48 > 0$ and

  $$\frac{\partial^2 z}{\partial x^2} = 4 > 0,$$

  so f has a minimum value (of $z = -16$) at $(x, y) = (0, 2)$.
- At $(x, y) = (0, -2)$, $\Delta = 48 > 0$ and

  $$\frac{\partial^2 z}{\partial x^2} = -4 < 0,$$

  so f has a maximum value (of $z = 16$) at $(x, y) = (0, -2)$.

4. [avg: 2.2/10] A bullet of mass m, fired straight up with an initial velocity of v_0 at $t = 0$, is slowed down by the force of gravity and a drag force of air resistance kv^2, for some positive constant k. As the bullet moves upward, its velocity v satisfies the equation

$$m\frac{dv}{dt} = -(kv^2 + mg).$$

(a) [6 marks] Solve the above initial value problem for v as a function of t.

Solution: you could rearrange the given equation as

$$m\frac{dv}{dt} = -(kv^2 + mg) \Leftrightarrow \frac{dv}{dt} = -\frac{k}{m}\left(v^2 + \frac{m}{k}g\right).$$

Let $w = k/m$ and separate variables:

$$\int \frac{dv}{v^2 + \frac{g}{w}} = -w\int dt \Leftrightarrow \sqrt{\frac{w}{g}}\tan^{-1}\left(\sqrt{\frac{w}{g}}\,v\right) = -w\,t + C.$$

To find C use the initial condition: $v = v_0, t = 0$: $C = \sqrt{\frac{w}{g}}\tan^{-1}\left(\sqrt{\frac{w}{g}}\,v_0\right)$. Then

$$\begin{aligned}
\sqrt{\frac{w}{g}}\tan^{-1}\left(\sqrt{\frac{w}{g}}\,v\right) &= -w\,t + \sqrt{\frac{w}{g}}\tan^{-1}\left(\sqrt{\frac{w}{g}}\,v_0\right)\\
\Rightarrow \tan^{-1}\left(\sqrt{\frac{w}{g}}\,v\right) &= -\sqrt{gw}\,t + \tan^{-1}\left(\sqrt{\frac{w}{g}}\,v_0\right)
\end{aligned}$$

$$\Rightarrow v = \sqrt{\frac{g}{w}}\tan\left(-\sqrt{gw}\,t + \tan^{-1}\left(\sqrt{\frac{w}{g}}\,v_0\right)\right) = \sqrt{\frac{mg}{k}}\tan\left(\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) - \sqrt{\frac{kg}{m}}\,t\right)$$

(b) [4 marks] If the height of the bullet at time t is y and its initial height is 0 at $t = 0$, solve for y in terms of t.

Solution:

$$\begin{aligned}
v = \frac{dy}{dt} \quad &\Rightarrow \quad y = \int \sqrt{\frac{mg}{k}}\tan\left(\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) - \sqrt{\frac{kg}{m}}\,t\right)\,dt\\
&\Rightarrow \quad y = \sqrt{\frac{mg}{k}}\ln\left|\sec\left(\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) - \sqrt{\frac{kg}{m}}\,t\right)\right|\left(-\sqrt{\frac{m}{kg}}\right) + C
\end{aligned}$$

Since $y_0 = 0$, you can evaluate C to obtain:

$$y = \frac{m}{k}\ln\left|\cos\left(\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right) - \sqrt{\frac{kg}{m}}\,t\right)\right| + \frac{m}{k}\ln\left|\sec\left(\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)\right|$$

Aside: all of the above is only valid until the bullet comes to rest, ie. for

$$0 \le t \le \sqrt{\frac{m}{kg}}\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right).$$

In that time the bullet reaches a maximum height of $y = \frac{m}{k}\ln\left|\sec\left(\tan^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)\right|$

5 [avg: 7.7/12]

5.(a) [6 marks] Find the interval of convergence of the power series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-2)^k}{\sqrt{k}\,4^k}$.

Solution: let $u_k = (-1)^{k+1}/(\sqrt{k}\,4^k)$. The **radius of convergence** is

$$R = \lim_{k\to\infty}\left|\frac{u_k}{u_{k+1}}\right| = \lim_{k\to\infty}\frac{\sqrt{k+1}\,4^{k+1}}{\sqrt{k}\,4^k} = 4\lim_{k\to\infty}\sqrt{\frac{k+1}{k}} = 4,$$

so the **open** interval of convergence is $(-2,6)$. Next, **check the endpoints:**

- $x=-2 \Rightarrow$
$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-2)^k}{\sqrt{k}\,4^k} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(-1)^k}{\sqrt{k}} = -\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}},$$
which **diverges** because it is a p-series with $p=1/2<1$.
- $x=6 \Rightarrow$
$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-2)^k}{\sqrt{k}\,4^k} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}},$$
which **converges** by the alternating series test.

Answer: The interval of convergence is $(-2,6]$.

5.(b) [6 marks] Use series to find an approximation of $\displaystyle\int_0^{1/2}\frac{dx}{(1+x^4)^{3/2}}$ correct to within 10^{-4}.

Solution: use the binomial series, and integrate term by term:

$$\begin{aligned}
\int_0^{1/2}\frac{dx}{(1+x^4)^{3/2}} &= \int_0^{1/2}(1+x^4)^{-3/2}\,dx \\
&= \int_0^{1/2}\left(1-\frac{3\,x^4}{2}+\frac{15\,x^8}{8}-\frac{35\,x^{12}}{16}+\cdots\right)dx \\
&= \left[x-\frac{3\,x^5}{10}+\frac{5\,x^9}{24}-\frac{35\,x^{13}}{208}+\cdots\right]_0^{1/2} \\
&= \underbrace{\frac{1}{2}-\frac{3}{320}+\frac{5}{12288}}-\frac{35}{(208)(2^{13})}+\cdots \\
&\approx 0.4910319010,
\end{aligned}$$

correct to within

$$\frac{35}{(208)(2^{13})}\approx 2.0\times 10^{-5} < 10^{-4},$$

by the alternating series test remainder term. Must quote this to get full marks!

6 [avg: 7.7/13]

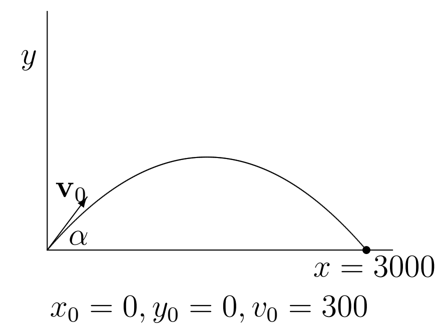
6.(a) [7 marks] Find two elevation angles that will enable a shell, fired from ground level with a muzzle speed of 300 m/sec, to hit a ground-level target 3 km away. (Ignore air resistance; use $g = 9.8$ m/s^2.)

Solution: let $x_0 = 0, y_0 = 0$ be the initial position from where the shell is fired; let t be measured in seconds since the instant the shell is fired; let the angle to the horizonal at which the shell is fired be α. We are given initial speed $v_0 = 300$. The parametric equations of motion for the shell are

$$x = 300(\cos\alpha)t \ , \ y = 300(\sin\alpha)t - \frac{1}{2}gt^2.$$

If the trajectory of the shell is to pass through the point $(x, y) = (3000, 0)$, we must have:

$$\begin{cases} 300\,t\cos\alpha & = & 3000 \\ 300\,t\,\sin\alpha - \frac{1}{2}g\,t^2 & = & 0 \end{cases}$$



Thus

$$t = 0 \text{ or } t = \frac{600\sin\alpha}{g}.$$

Substitute the non-zero value of t into the first equation and use the appropriate double angle formula:

$$300\left(\frac{600\sin\alpha}{g}\right)\cos\alpha = 3000 \Leftrightarrow \sin(2\alpha) = \frac{g}{30} \Rightarrow 2\alpha = \sin^{-1}\left(\frac{g}{30}\right)$$
$$\Rightarrow 2\alpha \approx 19^\circ \text{ or } 161^\circ \Leftrightarrow \alpha \approx 9.5^\circ \text{ or } 80.5^\circ.$$

6.(b) [6 marks] What is the $10th$ Maclaurin polynomial[1] of $f(x) = e^{-x^2}\ln(1 + x^4)$?

Solution: use the series for e^z with $z = -x^2$ and the series for $\ln(1 + w)$ with $w = x^4$:

$$\begin{aligned} e^{-x^2}\ln(1+x^4) &= \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots\right)\left(x^4 - \frac{x^8}{2} + \cdots\right) \\ &= x^4 - \frac{x^8}{2} - x^6 + \frac{x^{10}}{2} + \frac{x^8}{2} - \frac{x^{12}}{4} - \frac{x^{10}}{6} + \frac{x^{14}}{12} + \cdots \\ &= x^4 - x^6 + \frac{x^{10}}{3} + \cdots \end{aligned}$$

So the $10th$ Maclaurin polynomial of $f(x)$ is

$$P_{10}(x) = x^4 - x^6 + \frac{x^{10}}{3}.$$

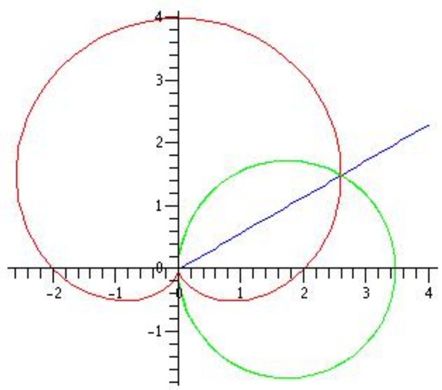
[1]same as the $10th$ degree Maclaurin polynomial

7. [avg: 7.3/13] Plot the two polar curves with polar equations

$$r = 2\sqrt{3}\,\cos\theta \text{ and } r = 2 + 2\,\sin\theta$$

and then find the area of the region that is common to the two curves. (If you can't find the intersection angle in the first quadrant, call it α and give your answer in terms of α.)

Solution: in the diagram below, the red curve is the polar graph of the cardioid



$$r_1 = 2 + 2\sin\theta,$$

and the green curve is the polar graph of the circle

$$r_2 = 2\sqrt{3}\,\cos\theta.$$

The two curves intersect (see below) at $\theta = \pi/6$, as indicated by the blue line, and at $\theta = -\pi/2$, for which both curves pass through the origin. Note that the area within the green circle is

$$\pi(\sqrt{3})^2 = 3\pi.$$

There are two ways you can calculate the required area:

$$A_1 = \frac{1}{2}\int_{-\pi/2}^{\pi/6} r_1^2\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} r_2^2\,d\theta \text{ or } A_2 = 3\pi - \frac{1}{2}\int_{-\pi/2}^{\pi/6}(r_2^2 - r_1^2)\,d\theta.$$

Calculation of A_2: the area in the green circle minus the area outside the red cardioid.

$$\begin{aligned}
A_2 = 3\pi - \frac{1}{2}\int_{-\pi/2}^{\pi/6}(r_2^2 - r_1^2)\,d\theta &= 3\pi - \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left(12\cos^2\theta - 4 - 8\sin\theta - 4\sin^2\theta\right)\,d\theta \\
&= 3\pi - \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left(8 - 8\sin\theta - 16\sin^2\theta\right)\,d\theta \\
&= 3\pi - \frac{1}{2}\int_{-\pi/2}^{\pi/6}(8 - 8\sin\theta - 8(1 - \cos(2\theta))\,d\theta \\
&= 3\pi - \frac{1}{2}\int_{-\pi/2}^{\pi/6}(8\cos(2\theta) - 8\sin\theta)\,d\theta \\
&= 3\pi - \frac{1}{2}\left[4\sin(2\theta) + 8\cos\theta\right]_{-\pi/2}^{\pi/6} = 3\pi - 3\sqrt{3}
\end{aligned}$$

Intersection Points: $r_1 = r_2 \Rightarrow r_1^2 = r_2^2 \Rightarrow 4 + 8\sin\theta + 4\sin^2\theta = 12\cos^2\theta = 12(1 - \sin^2\theta)$ Simplifying gives $2\sin^2\theta + \sin\theta - 1 = 0 \Leftrightarrow \sin\theta = 1/2$ or -1. So $\theta = \pi/6$ or $-\pi/2$. OR:

$$2\sqrt{3}\cos\theta = 2 + 2\sin\theta \Leftrightarrow \sqrt{3}\cos\theta - \sin\theta = 1 \Leftrightarrow \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}$$

$$\Leftrightarrow \cos(\pi/6 + \theta) = \frac{1}{2} \Rightarrow \frac{\pi}{6} + \theta = \pm\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6} \text{ or } -\frac{\pi}{2}.$$

8. [avg: 9.7/12] Consider the curve with vector equation $\mathbf{r} = \sin(e^t)\,\mathbf{i} + \cos(e^t)\,\mathbf{j} + \sqrt{3}\,e^t\,\mathbf{k}$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = e^t\cos(e^t)\,\mathbf{i} - e^t\sin(e^t)\,\mathbf{j} + \sqrt{3}\,e^t\,\mathbf{k};$$

$$\begin{aligned}\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{e^{2t}\cos^2(e^t) + e^{2t}\sin^2(e^t) + 3e^{2t}} \\ &= \sqrt{4e^{2t}} \\ &= 2\,e^t\end{aligned}$$

(b) [6 marks] Find an arc length parameterization of the curve, with reference point $(0,1,0)$, for which $t = -\infty$.

Solution:

$$s = \int_{-\infty}^{t}\left\|\frac{d\mathbf{r}}{du}\right\|du = \int_{-\infty}^{t} 2e^u\,du = \lim_{a\to-\infty}\,[2e^u]_a^t = 2e^t - 0 \Rightarrow e^t = \frac{s}{2}.$$

So the arc length parameterization of the curve is

$$\mathbf{r} = \sin(s/2)\,\mathbf{i} + \cos(s/2)\,\mathbf{j} + \sqrt{3}\,s/2\,\mathbf{k}.$$