University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2013**
First Year - CHE, CIV, IND, LME, MEC, MMS

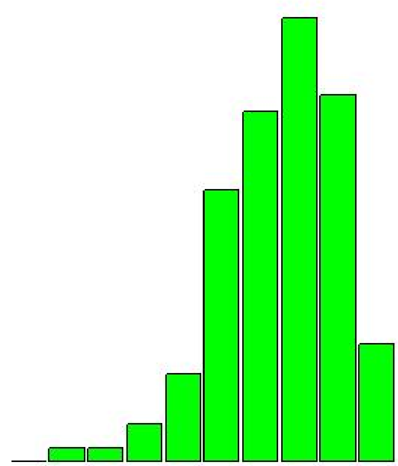
MAT187H1S - CALCULUS II
Exam Type: A
Duration: 150 min.

General Comments:

1. Eight questions were completely straightforward; only Question 8 was not.
2. The range on each question was zero to perfect; only Question 8 had a failing average.
3. This exam had the highest average on any MAT187H1S final exam that I have ever seen. It was probably too easy, or marked too leniently, or a combination of both. However, combined with the relatively low average term mark, the final average in the course is almost exactly the same as last year.
4. The algebra was atrocious in Question 1. Many students took only positive square roots, thereby throwing away half the critical points; many students thought $xy = 0$ implies $x = 0$ **and** $y = 0$. There were many miscalculations with the partial derivatives too!
5. Question 8 was the most challenging question and had the lowest average. Five solutions for Question 8(b) are given; the two shortest ones are included below, and the other three are at the end, on extra pages. Only two students got this question completely perfect.

Breakdown of Results: 506 students wrote this exam. The marks ranged from 15% to 97%, and the average mark was 69.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.9%
A	28.4%	80-89%	21.5%
B	26.1%	70-79%	26.1%
C	20.6%	60-69%	20.6%
D	16.0%	50-59%	16.0%
F	8.9%	40-49%	5.1%
		30-39%	2.2%
		20-29%	0.8%
		10-19%	0.8%
		0-9%	0.0%



1. [12 marks; avg: 8.77] Find all the critical points of the function $f(x, y) = 4x - 3x^3 - 2xy^2$, and determine if they are maximum points, minimum points, or saddle points.

Solution: Let $z = f(x, y)$. Then

$$\frac{\partial z}{\partial x} = 4 - 9x^2 - 2y^2 \text{ and } \frac{\partial z}{\partial y} = -4xy.$$

Critical points:

$$\left.\begin{array}{rcl} 4 - 9x^2 - 2y^2 & = & 0 \\ -4xy & = & 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 9x^2 + 2y^2 & = & 4 \\ xy & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} x \text{ or } y = 0 \\ \text{and} \\ 9x^2 + 2y^2 = 4 \end{array}\right.$$

So there are only four critical points, namely the intercepts of the ellipse:

$$(x, y) = (0, \sqrt{2}), (0, -\sqrt{2}), (2/3, 0) \text{ or } (-2/3, 0).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = -18x,\ \frac{\partial^2 z}{\partial y^2} = -4x,\ \frac{\partial^2 z}{\partial x \partial y} = -4y$$

and so

$$\Delta = (-18x)(-4x) - (-4y)^2 = 72x^2 - 16y^2.$$

- At each of $(x, y) = (0, \sqrt{2})$ or $(0, -\sqrt{2})$

  $$\Delta = -32 < 0,$$

  so f has a saddle point at each of $(x, y, z) = (0, \sqrt{2}, 0)$ and $(0, -\sqrt{2}, 0)$.
- At $(x, y) = (2/3, 0)$, $\Delta = 32 > 0$ and

  $$\frac{\partial^2 z}{\partial x^2} = -12 < 0,$$

  so f has a maximum value (of $z = 16/9$) at $(x, y) = (2/3, 0)$.
- At $(x, y) = (-2/3, 0)$, $\Delta = 32 > 0$ and

  $$\frac{\partial^2 z}{\partial x^2} = 12 > 0,$$

  so f has a minimum value (of $z = -16/9$) at $(x, y) = (-2/3, 0)$.

2. [10 marks; avg: 6.9] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

for some positive constant k, where T is the temperature of an object at time t in a room with constant ambient temperature A.

Suppose a freshly baked cake is taken out of the oven at 9:30 AM with temperature 200 C, and put on a table in a room with constant air temperature 20 C. If the temperature of the cake is 100 C at 9:50 AM, when will the temperature of the cake be 50 C ?

Solution: let $A = 20$ and separate variables. You can assume that $T > 20$.

$$\int \frac{dT}{T - 20} = \int -k\,dt \Rightarrow \ln(T - 20) = -kt + C.$$

Note: there is no need to solve this formula for T in general, but you could if you wanted.

Let t be measured in minutes and choose $t = 0$ to correspond to 9:30 AM. To find C take $t = 0, T = 200$:

$$\ln(200 - 20) = -k(0) + C \Rightarrow C = \ln 180.$$

To find k take $t = 20, T = 100$ and use $C = \ln 180$:

$$\ln(100 - 20) = -20k + \ln 180 \Rightarrow 20k = \ln 180 - \ln 80 = \ln 2.25 \Rightarrow k = \frac{1}{20}\ln 2.25.$$

To answer the question, let $T = 50$ and solve for t :

$$\ln(50 - 20) = -\frac{\ln 2.25}{20}t + \ln 180 \Rightarrow \frac{\ln 2.25}{20}t = \ln 180 - \ln 30 = \ln 6 \Rightarrow t = \frac{20\ln 6}{\ln 2.25} \approx 44.2.$$

So the temperature of the cake will be 50 C at about 10:14:12 AM.

3. [12 marks; avg: 9.2] Find the general solution for each of the following differential equations:

(a) [7 marks] $\dfrac{dy}{dx} + \dfrac{x\,y}{1+x^2} = 4x$

Solution: use the method of the integrating factor.

$$\mu = e^{\int (x/(1+x^2)\,dx} = e^{(\ln(1+x^2))/2} = (1+x^2)^{1/2} = \sqrt{1+x^2};$$

so

$$\begin{aligned} y &= \frac{1}{\mu}\int 4x\,\mu\,dx \\ &= \frac{1}{\sqrt{1+x^2}}\left(\int 4x\sqrt{1+x^2}\,dx\right) \\ &= \frac{1}{\sqrt{1+x^2}}\left(\frac{4}{3}(1+x^2)^{3/2} + C\right) \\ &= \frac{4}{3}(1+x^2) + \frac{C}{\sqrt{1+x^2}} \end{aligned}$$

(b) [5 marks] $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$

Solution: the auxiliary quadratic is $r^2 + 6r + 9$ which is a perfect square and has only the single real root $r = -3$. So the general solution is

$$y = Ae^{-3x} + Bx\,e^{-3x}.$$

4. [10 marks; avg: 8.1] Suppose $x = 16\cos t + 16t\sin t$ and $y = 16\sin t - 16t\cos t$ are the parametric equations of a curve. Find the following:

(a) [5 marks] all points on the curve, for $0 < t < 3\pi/2$, at which the tangent line to the curve is horizontal or vertical.

Solution:

$$\frac{dx}{dt} = -16\sin t + 16\sin t + 16t\cos t = 16t\cos t,$$
$$\frac{dy}{dt} = 16\cos t - 16\cos t + 16t\sin t = 16t\sin t.$$

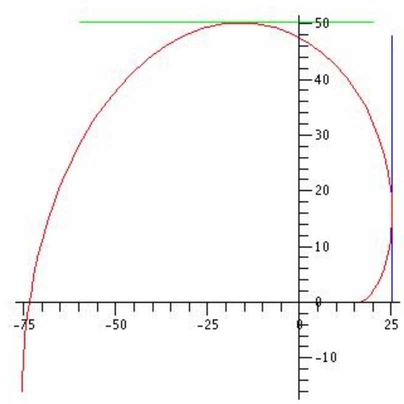
For horizontal tangents,

$$0 = \frac{dy}{dt} \Rightarrow t = \pi,$$

and the point is $(x, y) = (-16, 16\pi)$.
For vertical tangents,

$$0 = \frac{dx}{dt} \Rightarrow t = \frac{\pi}{2},$$

and the point is $(x, y) = (8\pi, 16)$.



(b) [5 marks] the length of the curve, for $0 \le t \le 3\pi/2$.

Solution: let L be the length of the specified curve.

$$\begin{aligned}
L &= \int_0^{3\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{3\pi/2} \sqrt{(16t\cos t)^2 + (16t\sin t)^2}\, dt \\
&= \int_0^{3\pi/2} \sqrt{16^2t^2\cos^2 t + 16^2t^2\sin^2 t}\, dt \\
&= \int_0^{3\pi/2} \sqrt{16^2t^2}\, dt \\
&= \int_0^{3\pi/2} 16t\, dt \\
&= \left[8t^2\right]_0^{3\pi/2} \\
&= 18\,\pi^2
\end{aligned}$$

5 [12 marks; avg: 8.87]

5.(a) [6 marks] Find the interval of convergence of the power series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k\tan^{-1}k}$.

Solution: the **radius of convergence** is

$$R=\lim_{k\to\infty}\left|\frac{u_k}{u_{k+1}}\right|=\lim_{k\to\infty}\frac{k+1}{k}\frac{\tan^{-1}(k+1)}{\tan^{-1}k}=(1)\frac{\pi/2}{\pi/2}=1,$$

so the open interval of convergence is $(-1,1)$. Next, **check the endpoints:**

- $x=-1\Rightarrow$
$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k\tan^{-1}k}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(-1)^k}{k\tan^{-1}k}=-\sum_{k=1}^{\infty}\frac{1}{k\tan^{-1}k},$$
which **diverges** by the limit comparison test: take $a_k=1/(k\tan^{-1}k)$ and $b_k=1/k$. Then $\sum b_k$ diverges, since it is the harmonic series, and
$$0<\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{1}{\tan^{-1}k}=\frac{2}{\pi}<\infty.$$
- $x=1\Rightarrow$
$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k\tan^{-1}k}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\tan^{-1}k}=\sum_{k=1}^{\infty}(-1)^{k+1}a_k,$$
which **converges** by the alternating series test: $a_k\to 0$ as $k\to\infty$, since $k\tan^{-1}k\to\infty$ as $k\to\infty$; and $a_{k+1}<a_k$ since $(k+1)\tan^{-1}(k+1)>k\tan^{-1}k$, as both k and $\tan^{-1}k$ are increasing functions of k, for $k\geq 1$.

Answer: the interval of convergence is $(-1,1]$.

5.(b) [6 marks] Use series to find an approximation of $\displaystyle\int_0^{1/2}\frac{dx}{\sqrt{1+x^4}}$ correct to within 10^{-5}.

Solution: use the binomial series, and integrate term by term:

$$\begin{aligned}\int_0^{1/2}\frac{dx}{\sqrt{1+x^4}} &= \int_0^{1/2}(1+x^4)^{-1/2}\,dx\\ &= \int_0^{1/2}\left(1-\frac{x^4}{2}+\frac{3x^8}{8}-\frac{5x^{12}}{16}+\cdots\right)dx\\ &= \left[x-\frac{x^5}{10}+\frac{x^9}{24}-\frac{5x^{13}}{208}+\cdots\right]_0^{1/2}\\ &= \underbrace{\frac{1}{2}-\frac{1}{320}+\frac{1}{12288}}-\frac{5}{(208)(2^{13})}+\cdots\\ &\approx 0.4969563802,\ \text{correct to within}\ \frac{5}{(208)(2^{13})}\approx 2.9\times 10^{-6}<10^{-5},\end{aligned}$$

by the alternating series test remainder term.

6. [10 marks; avg: 7.5] What is the initial velocity vector of a bullet if it is fired from ground level and hits a target 1 second later, given that the target is 150 m above ground level on a tower 300 m away as measured along the ground? Ignore air resistance; use $g = 9.8\ \mathrm{m/s^2}$.

Solution: let $x_0 = 0, y_0 = 0$ be the initial position from where the bullet is fired. Let v_0 be its initial speed, let the angle to the horizonal at which the bullet is fired be α. Let t be measured in seconds since the instant the bullet is fired. Then the parametric equations of motion for the bullet are

$$x = v_0(\cos\alpha)t \text{ and } y = v_0(\sin\alpha)t - \frac{1}{2}gt^2.$$

At $t = 1$ the bullet hits the target, so $(x, y) = (300, 150)$, giving us two equations to solve for v_0 and α:

$$v_0\cos\alpha = 300 \text{ and } v_0\sin\alpha = 150 + \frac{1}{2}g.$$

Dividing these two equations we get

$$\tan\alpha = \frac{150 + g/2}{300} = \frac{154.9}{300} = 0.516\overline{3} \Rightarrow \alpha \approx 27.31°.$$

Then

$$v_0 = \frac{300}{\cos\alpha} \approx 337.63.$$

Thus the initial speed of the bullet is 337.63 m/sec, or approximately 1,215.5 km/hr, and the direction of the initial velocity is approximately 27.31° above the horizontal.

Note: no calculus was actually performed in the above solution, since these equations of motion are straight physics, and you can use what you already know from physics. You could of course use calculus to derive these equations:

$$\mathbf{v} = \int \mathbf{a}\,dt = \int -g\,dt\,\mathbf{j} = -gt\,\mathbf{j} + \mathbf{v_0} = -gt\,\mathbf{j} + v_0\cos\alpha\,\mathbf{i} + v_0\sin\alpha\,\mathbf{j},$$

$$\mathbf{r} = \int \mathbf{v}\,dt = \int v_0\cos\alpha\,dt\,\mathbf{i} + \int (v_0\sin\alpha - gt)\,dt\,\mathbf{j} = (v_0\cos\alpha)t\,\mathbf{i} + \left((v_0\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{r_0},$$

whence

$$x = (v_0\cos\alpha)t \text{ and } y = (v_0\sin\alpha)t - \frac{1}{2}gt^2,$$

since $\mathbf{r_0} = 0\,\mathbf{i} + 0\,\mathbf{j}$.

Comment: the angle of inclination α is just slightly above the direct line of sight to the target, $\beta = \tan^{-1} 0.5$, since the bullet is going very fast, and the deflection caused by gravity in one second is only 4.9 m. This is another way to get α : aim at a point 4.9 m above the target so that

$$\tan\alpha = \frac{154.9}{300}.$$

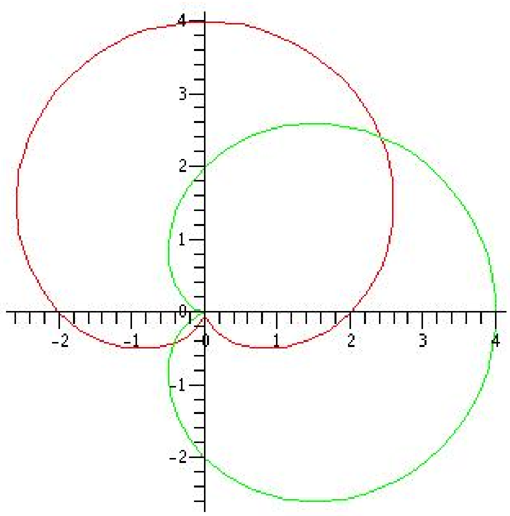
7. [12 marks; avg: 8.1] For $0 \le \theta \le 2\pi$, sketch the curves r_1, r_2 with polar equations

$$r_1 = 2 + 2\sin\theta;\ \ r_2 = 2 + 2\cos\theta,$$

and then find the area of the region outside the curve r_1 but inside the curve r_2.

Solution: at the intersection,

$$r_1 = r_2 \Leftrightarrow \sin\theta = \cos\theta \Leftrightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \text{ or } -\frac{3\pi}{4}.$$



In the diagram to the left, the red curve is the polar graph of

$$r_1 = 2 + 2\sin\theta,$$

and the green curve is the polar graph of

$$r_2 = 2 + 2\cos\theta.$$

The region in question is inside the green cardioid and outside the red cardioid. Its area is given by

$$A = \frac{1}{2}\int_{-3\pi/4}^{\pi/4} (r_2^2 - r_1^2)\, d\theta$$

To integrate, you can use the double angle formula $\cos(2\theta) = \cos^2\theta - \sin^2\theta$. We have

$$\begin{aligned}
A = \frac{1}{2}\int_{-3\pi/4}^{\pi/4} (r_2^2 - r_1^2)\, d\theta &= \frac{1}{2}\int_{-3\pi/4}^{\pi/4} \left(4 + 8\cos\theta + 4\cos^2\theta - 4 - 8\sin\theta - 4\sin^2\theta\right) d\theta \\
&= \frac{1}{2}\int_{-3\pi/4}^{\pi/4} \left(8\cos\theta - 8\sin\theta + 4\cos^2\theta - 4\sin^2\theta\right) d\theta \\
&= \int_{-3\pi/4}^{\pi/4} \left(4\cos\theta - 4\sin\theta + 2\cos(2\theta)\right) d\theta \\
&= \left[4\sin\theta + 4\cos\theta + \sin(2\theta)\right]_{-3\pi/4}^{\pi/4} \\
&= 8\sqrt{2}
\end{aligned}$$

8 [10 marks: avg: 1.9]

8.(a) [5 marks] Let $f(x) = 1 - x - x^2 + x^3 - x^4 - x^5 + x^6 - x^7 - x^8 + x^9 - x^{10} - x^{11} + \cdots$.
Show that this series converges absolutely for $|x| < 1$ and then compute its sum.
Hint: write $f(x)$ as a sum of three infinite geometric series with common ratio x^3.

Solution: let a_n be the coefficient of x^n in $f(x)$. Then $|a_n| = 1$, and the radius of convergence of $f(x)$ is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = 1,$$

which means the series for $f(x)$ converges absolutely if $|x| < 1$. **Or** use the ratio test for absolute convergence:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x| = |x|,$$

and the result follows. As for the sum: using the hint,

$$\begin{aligned} f(x) &= 1 + x^3 + x^6 + \cdots - (x + x^4 + x^7 + \cdots) - (x^2 + x^5 + x^8 + \cdots) \\ &= 1 + x^3 + x^6 + \cdots - x(1 + x^3 + x^6 + \cdots) - x^2(1 + x^3 + x^6 + \cdots) \\ &= \frac{1}{1-x^3} - \frac{x}{1-x^3} - \frac{x^2}{1-x^3} \\ &= \frac{1-x-x^2}{1-x^3} \end{aligned}$$

8.(b) [5 marks] Find the Maclaurin series of $g(x) = \dfrac{1+2x}{1+x+x^2}$.

Solution: use $1 - x^3 = (1 + x + x^2)(1 - x)$; then

$$\begin{aligned} g(x) = \frac{1+2x}{1+x+x^2} &= \frac{(1+2x)(1-x)}{1-x^3} \\ &= (1 + x - 2x^2)(1 + x^3 + x^6 + x^9 + \cdots) \\ &= 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \cdots \end{aligned}$$

Alternate Solution: use $(1 + z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots$ with $z = x + x^2$. Then

$$\begin{aligned} g(x) &= \frac{1+2x}{1+x+x^2} \\ &= (1+2x)\left(1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + (x+x^2)^4 - \cdots\right) \\ &= (1+2x)\left(1 - x - x^2 + x^2 + 2x^3 + x^4 - x^3 - 3x^4 - 3x^5 - x^6 + x^4 + 4x^5 \cdots\right) \\ &= (1+2x)\left(1 - x + x^3 - x^4 + x^5 + \cdots\right) \\ &= 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + \cdots \end{aligned}$$

9. [12 marks; avg: 10.4] Consider the helix with vector equation $\mathbf{r} = \cos(4t)\,\mathbf{i} + \sin(4t)\,\mathbf{j} + 3t\,\mathbf{k}$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = -4\sin(4t)\,\mathbf{i} + 4\cos(4t)\,\mathbf{j} + 3\,\mathbf{k};$$

$$\begin{aligned}\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{16\sin^2(4t) + 16\cos^2(4t) + 9} \\ &= \sqrt{16+9} \\ &= 5\end{aligned}$$

(b) [6 marks] Find an arc length parameterization of the curve, with reference point $(1, 0, 0)$, for which $t = 0$.

Solution:

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \int_0^t 5\,du = 5t \Rightarrow t = \frac{s}{5},$$

So the arc length parameterization of the curve is

$$\mathbf{r} = \cos(4s/5)\,\mathbf{i} + \sin(4s/5)\,\mathbf{j} + 3s/5\,\mathbf{k}.$$

Alternate Solution to Question 8(b): Some students tried to use partial fractions. This would work, but you have to faith in your calculations!

$$x^2 + x + 1 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{3}\,i}{2};$$

Let

$$\alpha = \frac{-1 + \sqrt{3}\,i}{2},\ \beta = \frac{-1 - \sqrt{3}\,i}{2}.$$

Then $\alpha\beta = 1,\ \alpha + \beta = -1,\ \alpha^2 = \beta,\ \beta^2 = \alpha,\ \alpha^3 = \beta^3 = 1$ and

$$\frac{1+2x}{1+x+x^2} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} \Rightarrow A = B = 1.$$

So

$$\begin{aligned}
\frac{1+2x}{1+x+x^2} &= \frac{1}{x-\alpha} + \frac{1}{x-\beta} \\
&= -\frac{1}{\alpha}\left(\frac{1}{1-x/\alpha}\right) - \frac{1}{\beta}\left(\frac{1}{1-x/\beta}\right) \\
&= -\frac{1}{\alpha}\left(1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \frac{x^3}{\alpha^3} + \cdots\right) - \frac{1}{\beta}\left(1 + \frac{x}{\beta} + \frac{x^2}{\beta^2} + \frac{x^3}{\beta^3} + \cdots\right) \\
&= -\frac{\alpha+\beta}{\alpha\beta} - \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}x - \frac{\alpha^3+\beta^3}{\alpha^3\beta^3}x^2 - \frac{\alpha^4+\beta^4}{\alpha^4\beta^4}x^3 + \cdots \\
&= 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + \cdots
\end{aligned}$$

Of course, we wouldn't expect anybody to do it this way, but it does work!

Alternate Solution to Question 8(b): observe that $g(x) = \dfrac{d\ln(1+x+x^2)}{dx}$ and use the series for $\ln(1+z)$. Then

$$\begin{aligned}
g(x) &= \frac{d\ln(1+x+x^2)}{dx} \\
&= \frac{d}{dx}\left((x+x^2) - \frac{(x+x^2)^2}{2} + \frac{(x+x^2)^3}{3} - \frac{(x+x^2)^4}{4} + \cdots\right) \\
&= 1 + 2x - (x+x^2)(1+2x) + (x+x^2)^2(1+2x) - (x+x^2)^3(1+2x) + \cdots \\
&= (1+2x)\left(1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + (x+x^2)^4 \cdots\right) \\
&= (1+2x)\left(1 - x - x^2 + x^2 + 2x^3 + x^4 - x^3 - 3x^4 - 3x^5 - x^6 + x^4 + 4x^5 \cdots\right) \\
&= (1+2x)\left(1 - x + x^3 - x^4 + x^5 + \cdots\right) \\
&= 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + \cdots
\end{aligned}$$

Alternate Solution to Question 8(b): simply use long division.

$$
\begin{array}{r|rrrrrrrrrrrrrrrrr}
 & 1 & + & x & - & 2x^2 & + & x^3 & + & x^4 & - & 2x^5 & + & \cdots \\
\hline
1+x+x^2 & 1 & + & 2x & & & & & & & & & & \\
 & 1 & + & x & + & x^2 & & & & & & & & \\
\cline{2-6}
 & & & x & - & x^2 & & & & & & & & \\
 & & & x & + & x^2 & + & x^3 & & & & & & \\
\cline{4-8}
 & & & & - & 2x^2 & - & x^3 & & & & & & \\
 & & & & - & 2x^2 & - & 2x^3 & - & 2x^4 & & & & \\
\cline{5-10}
 & & & & & & & x^3 & + & 2x^4 & & & & \\
 & & & & & & & x^3 & + & x^4 & + & x^5 & & \\
\cline{7-12}
 & & & & & & & & & x^4 & - & x^5 & & \\
 & & & & & & & & & x^4 & + & x^5 & + & x^6 \\
\cline{9-14}
 & & & & & & & & & & - & 2x^5 & - & x^6 & & \\
 & & & & & & & & & & - & 2x^5 & - & 2x^6 & - & 2x^7 \\
\cline{11-16}
 & & & & & & & & & & & & & x^6 & + & 2x^7 \\
 & & & & & & & & & & & & & & \cdots & \\
\end{array}
$$

So

$$
g(x) = \frac{1+2x}{1+x+x^2} = 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + \cdots
$$