University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2012**
First Year - CHE, CIV, IND, LME, MEC, MMS

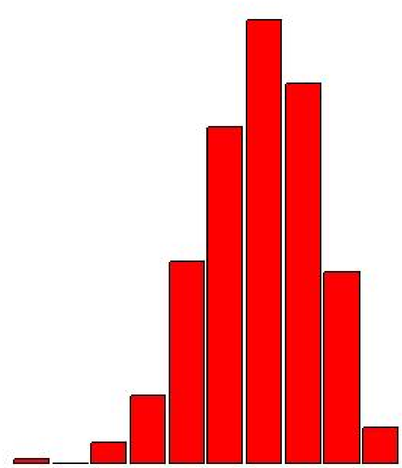
MAT187H1S - CALCULUS II
Exam Type: A
Duration: 150 min.

General Comments: some more comments can be found after Question 8.

1. A sample of common blunders: $\cos^6 t + \sin^6 t = 1$, $\sqrt{a^2+b^2} = a+b$, $\sin^3(\sin^{-1} x) = \sin^2 x$, $\int x^{-x}\,dx = x^{-x+1}$.
2. Questions 1, 2(a), 3, 4, 5, 6(a) and 8 are all considered routine–that's 76% of the exam.
3. Question 2(b) is best done by first finding the Maclaurin series of $f(x)$ and then truncating at degree 8.
4. In Question 3(b) you have to employ the remainder formula for alternating series to get full marks.
5. In Question 6 you must set up your solution so that the reader knows what $\mathbf{i}, \mathbf{j}, \mathbf{k}$ represent.
6. Question 7 is the most challenging question on the exam. But 7(a) requires only that you use the given reduction formula; everybody should have gotten it, but very few did!

Breakdown of Results: 474 students wrote this exam. The marks ranged from 6% to 98%, and the average mark was 63.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.1%
A	13.5%	80-89%	11.4%
B	22.6%	70-79%	22.6%
C	26.4%	60-69%	26.4%
D	20.0%	50-59%	20.0%
F	17.5%	40-49%	12.0%
		30-39%	4.0%
		20-29%	1.3%
		10-19%	0.0%
		0-9%	0.2%



1. [15 marks; avg: 12.9] Find the general solution, y as a function of x, for each of the following differential equations:

   (a) [4 marks] $\frac{dy}{dx} = \frac{1}{3xy^2}$

   Solution: separate variables. For $x \neq 0$:

$$\int 3y^2\,dy = \int \frac{dx}{x} \Rightarrow y^3 = \ln|x| + C \Rightarrow y = (\ln|x| + C)^{1/3}.$$

   (b) [6 marks] $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^3+4}$

   Solution: use the method of the integrating factor.

$$\mu = e^{\int (2/x)\,dx} = e^{2\ln|x|} = |x|^2 = x^2;$$

   so

$$\begin{aligned} y &= \frac{1}{\mu}\int \frac{\mu}{x^3+4}\,dx \\ &= \frac{1}{x^2}\left(\int \frac{x^2}{x^3+4}\,dx\right) \\ &= \frac{1}{x^2}\left(\frac{1}{3}\ln|x^3+4| + C\right) \end{aligned}$$

   (c) [5 marks] $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 8y = 0$

   Solution: the auxiliary quadratic is $r^2 + 6r + 8$ which has roots $r = -2, -4$. So the general solution is

$$y = Ae^{-2x} + Be^{-4x}.$$

2. [10 marks; avg: 6.1] Find the following:

(a) [5 marks] the interval of convergence of the power series $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{(x-3)^k}{\sqrt{k+1}}$.

Solution: the radius of convergence is $R=\displaystyle\lim_{k\to\infty}\left|\frac{u_k}{u_{k+1}}\right|=\lim_{k\to\infty}\frac{\sqrt{k+2}}{\sqrt{k+1}}=1$, so the open interval of convergence is $(-1+3,3+1)=(2,4)$. Now check the endpoints:

$$x=2\Rightarrow\sum_{k=0}^{\infty}(-1)^k\frac{(x-3)^k}{\sqrt{k+1}}=\sum_{k=0}^{\infty}(-1)^k\frac{(-1)^k}{\sqrt{k+1}}=\sum_{k=0}^{\infty}\frac{1}{\sqrt{k+1}},$$

which diverges, by the integral test.

$$x=4\Rightarrow\sum_{k=0}^{\infty}(-1)^k\frac{(x-3)^k}{\sqrt{k+1}}=\sum_{k=0}^{\infty}(-1)^k\frac{1}{\sqrt{k+1}},$$

which converges, by the alternating series test.
Answer: the interval of convergence is $(2,4]$.

(b) [5 marks] the $8th$ ($=8th$ degree) Maclaurin polynomial of $f(x)=\dfrac{x^2}{(1+x^2)^{2/3}}$.

Solution: use the binomial series

$$(1+z)^\alpha=1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\cdots,$$

with $z=x^2,\alpha=-2/3$. Then:

$$\begin{aligned}&\frac{x^2}{(1+x^2)^{2/3}}\\ =\;& x^2\left(1-\frac{2}{3}x^2+\frac{(-2/3)(-2/3-1)}{2!}x^4+\frac{(-2/3)(-2/3-1)(-2/3-2)}{3!}x^6+\cdots\right)\\ =\;& x^2\left(1-\frac{2}{3}x^2+\frac{5}{9}x^4-\frac{40}{81}x^6+\cdots\right)\\ =\;& x^2-\frac{2}{3}x^4+\frac{5}{9}x^6-\frac{40}{81}x^8+\cdots\end{aligned}$$

So

$$P_8(x)=x^2-\frac{2}{3}x^4+\frac{5}{9}x^6-\frac{40}{81}x^8.$$

3. [15 marks; avg: 11.3]

3.(a) [8 marks] Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(x, y) = (\pi/2, 1)$ if $x = t - \cos t, y = 1 + \sin(2t)$.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(2t)}{1 + \sin t};$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{-4\sin(2t)\,(1 + \sin t) - 2\cos(2t)(-\cos t)}{(1 + \sin t)^3}.$$

At $(x, y) = (\pi/2, 1), t = \pi/2$ so

$$\frac{dy}{dx} = -1 \text{ and } \frac{d^2y}{dx^2} = 0.$$

3.(b) [7 marks] Approximate $\displaystyle\int_0^1 \sin(x^4)\,dx$ correct to within 10^{-5}, and explain how you know your answer is correct to within 10^{-5}.

Solution: use the series $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots$ with $z = x^4$:

$$\begin{aligned}
\int_0^1 \sin(x^4)\,dx &= \int_0^1 \left(x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \frac{x^{28}}{7!} + \cdots\right) dx \\
&= \left[\frac{x^5}{5} - \frac{x^{13}}{13(3!)} + \frac{x^{21}}{21(5!)} - \frac{x^{29}}{29(7!)} + \cdots\right]_0^1 \\
&= \frac{1}{5} - \frac{1}{13(3!)} + \frac{1}{21(5!)} - \frac{1}{29(7!)} + \cdots
\end{aligned}$$

Since this is an alternating series the sum of the first three terms is correct to within

$$\frac{1}{29(7!)} = 0.000006841\ldots < 10^{-5},$$

so

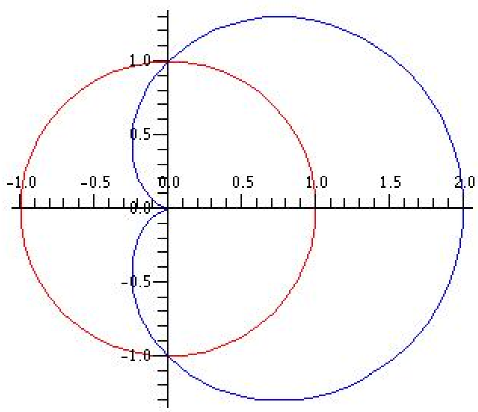
$$\int_0^1 \sin(x^4)\,dx = \frac{1}{5} - \frac{1}{13(3!)} + \frac{1}{21(5!)} = 0.187576312\ldots$$

correct to within $0.000006841\ldots < 10^{-5}$.

4. [10 marks; avg: 7.3] Plot both the circle with polar equation $r = 1$ and the cardioid with polar equation $r = 1 + \cos\theta$, in the same plane, and then find the area of the region that is inside the circle but outside the cardioid.

Solution: at the intersection,

$$1 = 1 + \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$



The region in question is inside the red circle and outside the blue cardioid. Its area is given by

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left(1^2 - (1+\cos\theta)^2\right)\,d\theta \\ &= \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left(-2\cos\theta - \cos^2\theta\right)\,d\theta \end{aligned}$$

To integrate, use the double angle formula $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$. Then

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left(-2\cos\theta - \cos^2\theta\right)\,d\theta \\ &= -\int_{\pi/2}^{3\pi/2} \left(\cos\theta + \frac{1}{4} + \frac{\cos(2\theta)}{4}\right)\,d\theta \\ &= -\left[\sin\theta + \frac{\theta}{4} + \frac{\sin(2\theta)}{8}\right]_{\pi/2}^{3\pi/2} \\ &= -\left[-2 + \frac{\pi}{4}\right] = 2 - \frac{\pi}{4} \end{aligned}$$

5. [13 marks; avg: 7.8] Find all the critical points of the function

$$f(x,y) = x^2y^2 - 2xy^2 + 3x^2y - 6xy,$$

and determine if they are maximum points, minimum points, or saddle points.

Solution: Let $z = f(x,y)$. Then

$$\frac{\partial z}{\partial x} = 2xy^2 - 2y^2 + 6xy - 6y \text{ and } \frac{\partial z}{\partial y} = 2x^2y - 4xy + 3x^2 - 6x.$$

Critical points:

$$\left.\begin{array}{rcl} 2xy^2 - 2y^2 + 6xy - 6y & = & 0 \\ 2x^2y - 4xy + 3x^2 - 6x & = & 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 2y(x-1)(y+3) & = & 0 \\ x(x-2)(2y+3) & = & 0 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{c} y = 0, x = 1 \text{ or } y = -3 \\ \text{and} \\ x = 0, x = 2 \text{ or } y = -3/2 \end{array}\right.$$

So there are only five critical points:

$$(x,y) = (0,0), (0,-3), (2,0), (2,-3) \text{ or } (1,-3/2).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 2y^2 + 6y,\ \frac{\partial^2 z}{\partial y^2} = 2x^2 - 4x,\ \frac{\partial^2 z}{\partial x \partial y} = 4xy - 4y + 6x - 6$$

and

$$\Delta = (2y^2 + 6y)(2x^2 - 4x) - (4xy - 4y + 6x - 6)^2.$$

At each of $(x,y) = (0,0), (0,-3), (2,0)$ or $(2,-3)$

$$\Delta = -36 < 0,$$

so f has a saddle point at each of $(x,y,z) = (0,0,0), (0,-3,0), (2,0,0)$ or $(2,-3,0)$.

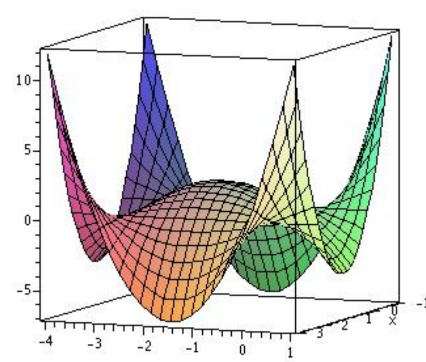
At $(x,y) = (1,-3/2)$, $\Delta = 9 > 0$ and

$$\frac{\partial^2 z}{\partial x^2} = -\frac{9}{2} < 0,$$

or use

$$\frac{\partial^2 z}{\partial y^2} = -2 < 0,$$

to conclude that f has a maximum value (of $z = 9/4$) at $(x,y) = (1,-3/2)$.



6. [12 marks; avg: 7.1] A small projectile is fired from ground level in an easterly direction with an initial speed of 300 m/s at an angle of 30° to the horizontal. A crosswind blows from south to north producing an acceleration of the projectile of 0.4 m/s^2 to the north. Assume the ground is flat, and let $g = 10$ m/s^2 (for easier calculations.)

(a) [6 marks] Where does the projectile land?

Solution: let $\mathbf{i}$ be due east, $\mathbf{j}$ due north, $\mathbf{k}$ straight up; let the initial position be $\mathbf{r}_0 = \langle 0, 0, 0\rangle$. We have $\mathbf{a} = 0.4\,\mathbf{j} - 10\,\mathbf{k}$ and

$$\mathbf{v}_0 = 300\cos 30°\,\mathbf{i} + 300\sin 30°\,\mathbf{k} = 150\sqrt{3}\,\mathbf{i} + 150\,\mathbf{k}.$$

Then

$$\mathbf{v} = \int \mathbf{a}\,dt = 0.4t\,\mathbf{j} - 10t\,\mathbf{k} + \mathbf{v}_0 = 150\sqrt{3}\,\mathbf{i} + 0.4t\,\mathbf{j} + (150 - 10t)\,\mathbf{k}$$

and

$$\mathbf{r} = \int \mathbf{v}\,dt = 150\sqrt{3}\,t\,\mathbf{i} + 0.2t^2\,\mathbf{j} + (150t - 5t^2)\,\mathbf{k},$$

since $\mathbf{r}_0 = \langle 0, 0, 0\rangle$. The projectile lands when $z = 0$, so $t = 30$. For $t = 30$,

$$x = 4500\sqrt{3} \text{ and } y = 180.$$

(b) [6 marks] In order to correct for the crosswind and make the projectile land due east of the launch site, at what angle from due east must the projectile be fired? Assume the same initial speed and angle of elevation as in part (a).

Solution: let the angle from due east be θ, then the initial velocity is

$$\mathbf{v}_0 = 150\sqrt{3}\cos\theta\,\mathbf{i} + 150\sqrt{3}\sin\theta\,\mathbf{j} + 150\,\mathbf{k}$$

and $\mathbf{a}$ is the same as in part (a). Integrating as before:

$$\mathbf{v} = 150\sqrt{3}\cos\theta\,\mathbf{i} + (150\sqrt{3}\sin\theta + 0.4t)\,\mathbf{j} + (150 - 10t)\,\mathbf{k},$$

$$\mathbf{r} = 150\sqrt{3}(\cos\theta)\,t\,\mathbf{i} + (150\sqrt{3}(\sin\theta)\,t + 0.2t^2)\,\mathbf{j} + (150t - 5t^2)\,\mathbf{k}.$$

As before it still takes $t = 30$ seconds to land, but now we want $y = 0$ at the landing point. So

$$150\sqrt{3}\,\sin\theta + 0.2(30) = 0 \Leftrightarrow \sin\theta = -\frac{1}{25\sqrt{3}} \Rightarrow \theta = \sin^{-1}\left(-\frac{1}{25\sqrt{3}}\right).$$

That is, $\theta \approx -1.3233°$ north of east, or 1.3233° south of east.

7. [13 marks; avg: 2.4]

7.(a) [6 marks] Use the reduction formula $\int_0^1 x^n (\ln x)^m \, dx = -\frac{m}{n+1} \int_0^1 x^n (\ln x)^{m-1} \, dx$ to show that

$$\int_0^1 x^n (\ln x)^n \, dx = (-1)^n \frac{n!}{(n+1)^{n+1}}.$$

Solution: start with $m = n$ and use the reduction formula n times:

$$\begin{aligned}
\int_0^1 x^n (\ln x)^n \, dx &= -\frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} \, dx \\
&= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+1}\right) \int_0^1 x^n (\ln x)^{n-2} \, dx \\
\text{(after } n \text{ times)} &= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+1}\right) \cdots \left(-\frac{2}{n+1}\right)\left(-\frac{1}{n+1}\right) \int_0^1 x^n \, dx \\
&= (-1)^n \frac{n!}{(n+1)^n} \left[\frac{x^{n+1}}{n+1}\right]_0^1 \\
&= (-1)^n \frac{n!}{(n+1)^{n+1}}
\end{aligned}$$

7.(b) [7 marks] Using part (a), show that $\int_0^1 \frac{dx}{x^x} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots$

Solution: use $x^{-x} = e^{-x \ln x}$ and then use the series for e^z in the integrand:

$$\begin{aligned}
\int_0^1 \frac{dx}{x^x} = \int_0^1 e^{-x \ln x} \, dx &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-x)^n (\ln x)^n}{n!}\right) dx \\
&= \sum_{n=0}^{\infty} \left(\int_0^1 \frac{(-x)^n (\ln x)^n}{n!} \, dx\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \int_0^1 x^n (\ln x)^n \, dx\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}}\right), \text{ by part(a)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} \\
&= \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} \cdots
\end{aligned}$$

Historical Note: this formula was proved by Johann Bernoulli in 1697.

8. [12 marks; avg: 8.7] Consider the curve $\mathbf{r} = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, for $0 \le t \le \pi/2$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = -3\cos^2 t \sin t\,\mathbf{i} + 3\sin^2 t \cos t\,\mathbf{j};$$

$$\begin{aligned}
\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \\
&= 3\sqrt{\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)} \\
&= 3\sqrt{\cos^2 t \sin^2 t} \\
&= 3\cos t \sin t,\ \text{ since } 0 \le t \le \pi/2
\end{aligned}$$

(b) [6 marks] Find an arc length parameterization of the curve, with reference point $(1, 0)$, for which $t = 0$.

Solution:

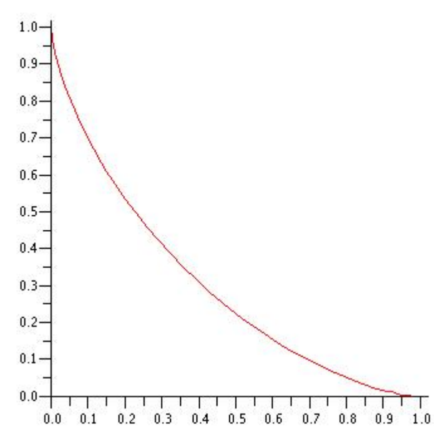
$$\begin{aligned}
s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du &= 3\int_0^t \cos u \sin u\, du \\
&= \frac{3}{2}\left[\sin^2 u\right]_0^t \\
&= \frac{3}{2}\sin^2 t \\
\Rightarrow \sin t &= \sqrt{\frac{2s}{3}},\ \text{ since } \sin t \ge 0 \text{ for } 0 \le t \le \pi/2 \\
\Rightarrow \cos t &= \sqrt{\frac{3-2s}{3}},\ \text{ since } \cos t \ge 0 \text{ for } 0 \le t \le \pi/2
\end{aligned}$$

So the arc length parameterization of the curve is

$$\mathbf{r} = \left(\frac{3-2s}{3}\right)^{3/2}\mathbf{i} + \left(\frac{2s}{3}\right)^{3/2}\mathbf{j},$$

for $0 \le s \le 3/2$. Out of interest, compare with its Cartesian equation:

$$x^{2/3} + y^{2/3} = 1.$$



More Detailed Comments, From the Markers:

Question 5: Very few students could find the critical points, let alone classify them, even though this question is standard and even though both partial derivatives factor completely. Some people could find the critical x and y values but many could not put them together properly (for instance $(2,0)$ and $(1,-3/2)$ are critical points but $(2,-3/2)$ is not). Performance was generally poor, but marking was generous: a student could earn 8 out of 13 just for finding and classifying just one of the 5 critical points correctly, assuming no arithmetic errors, and assuming the student did not generate too many other incorrect critical points, or other errors along the way. But still, many did not get 8 marks. Less than 20 students found all 5 critical points. Presentation was awful: only a fraction of students had the courtesy to write a sentence declaring "this is what I think the critical points are." Instead, x and y values appeared everywhere—sometimes with no obvious relation to each other.

Question 6: This question was done terribly in general. Most students treated this as a 2D motion problem instead of 3D, with the wind pointing into the sky. (N, E, S, W are directions in the plane, and the vector $\mathbf{k}$ is orthogonal to all of them!) Because of this, most students could not deal with part (b), because in their 2D version there was no room for another angle other than the given one. Even many students who realized it was a 3D problem made many numerical errors, some coming from not integrating $\mathbf{v}$ properly to get $\mathbf{r}$ (a common mistake was having $0.4t^2$ as one of the components instead of $0.2t^2$, and $10t^2$ instead of $5t^2$ in another.) Speaking of components: it was staggering how many students did not use $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (or some other bookkeeping system) to separate the components, leading to all kinds of confusion, like adding $\mathbf{j}$ and $\mathbf{k}$ components together. Presentation was poor: many statements appeared such as "the angle required is alpha due east." What does that even mean?

Question 7: For most students this question was a total write off. Many students tried to use integration by parts in part (a), but that is totally unnecessary because the required reduction formula is already given. Many students used the reduction formula (with $m = n$) once, then stopped. Those who continued using the reduction formula got at least 3 marks. To get full marks you had to explain how after n uses of the reduction formula, the remaining integral

$$\int_0^1 x^n \, dx = \frac{1}{n+1},$$

gives the final factor of $n + 1$ in the denominator. Only five students solved this question completely. I will write a good letter of reference for any one of them–DB.

Question 8: In part (a) many students integrated instead of differentiated; many rewrote $\cos^3 t$ and $\sin^3 t$ as $\cos t\,(1 + \cos(2t))/2$ and $\sin t\,(1 - \cos(2t))/2$, respectively, which only made the problem worse; many forgot to use the chain rule. In part (b) many students found the length of the curve, which is not what was asked for. Very few students bothered to simplify their final answer.