University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2011**
First Year - CHE, CIV, IND, LME, MEC, MMS

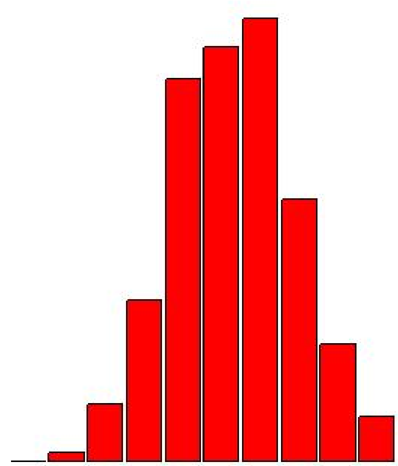
MAT187H1S - CALCULUS II
Exam Type: A
Duration: 150 min.

General Comments:

1. Questions 1, 2, 3, 6, 7 and 9 are all considered routine–some of them right out of the homework–and should all have been aced.
2. In Question 4 surprisingly few students could find the correct two intersection angles in the first quadrant. Worse: many students wrote $2\theta = \pi/6 \Rightarrow \theta = \pi/3$! Moreover, Question 4 is a routine one-step problem, but many students turned into a twisted multi-step computation.
3. Question 5 was supposed to be a challenge but many students solved it, using a variety of methods.
4. Question 8 caused a lot of problems! Apparently many students could not figure out what it meant. The actual math was short and easy.

Breakdown of Results: 469 students wrote this exam. The marks ranged from 11% to 98%, and the average mark was 57.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.3%
A	8.5%	80-89%	6.2%
B	13.9%	70-79%	13.9%
C	23.4%	60-69%	23.4%
D	22.0%	50-59%	22.0%
F	32.2%	40-49%	20.3%
		30-39%	8.5%
		20-29%	3.0%
		10-19%	0.4%
		0-9%	0.0%



1. Find the general solutions to the three following differential equations:

   (a) $\dfrac{dy}{dx} = \dfrac{1}{3xy^2}$

   Solution: separate variables. For $x \neq 0$:

$$\int 3y^2\,dy = \int \frac{dx}{x} \Rightarrow y^3 = \ln|x| + C \Rightarrow y = (\ln|x| + C)^{1/3}.$$

   (b) $\dfrac{dy}{dx} - y\,\tan x = \sin x$

   Solution: use the method of the integrating factor.

$$\mu = e^{-\int \tan x\,dx} = e^{-\ln|\sec x|} = |\cos x|;$$

   take $\mu = \cos x$ or $\mu = -\cos x$. Either way,

$$\begin{aligned} y &= \frac{1}{\mu}\int \mu\,dx \\ &= \sec x \left(\int \sin x\,\cos x\,dx\right) \\ &= \sec x \left(\frac{\sin^2 x}{2} + C\right) \\ &= \frac{1}{2}\tan x \sin x + C\,\sec x \end{aligned}$$

   (c) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 8y = 0$

   Solution: the auxiliary quadratic is $r^2 + 4r + 8$ which has roots $r = -2 \pm 2i$. So the general solution is

$$y = Ae^{-2x}\cos(2x) + Be^{-2x}\sin(2x).$$

2.(a) What is the interval of convergence of the power series $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{(x+5)^k}{k+2}$?

Solution: the radius of convergence is $R = \displaystyle\lim_{k\to\infty}\left|\frac{u_k}{u_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{(-1)^k(k+3)}{(-1)^{k+1}(k+2)}\right| = 1$, so the open interval of convergence is $(-1-5, -5+1) = (-6, -4)$. Now check the endpoints:

$$x = -6 \Rightarrow \sum_{k=0}^{\infty}(-1)^k \frac{(x+5)^k}{k+2} = \sum_{k=0}^{\infty}(-1)^k \frac{(-1)^k}{k+2} = \sum_{k=0}^{\infty} \frac{1}{k+2},$$

which diverges, by the integral test or by limit comparison with the harmonic series.

$$x = -4 \Rightarrow \sum_{k=0}^{\infty}(-1)^k \frac{(x+5)^k}{k+2} = \sum_{k=0}^{\infty}(-1)^k \frac{1}{k+2},$$

which converges, by the alternating series test. Answer: the interval of convergence is $(-6, -4]$.

2.(b) Find the first four non-zero terms of the Maclaurin series for $\dfrac{x^2}{x^2+1}$.

Solution: use $1 + z + z^2 + z^3 + \cdots + z^n + \cdots = \dfrac{1}{1-z}$, with $z = -x^2$. Then:

$$\frac{x^2}{x^2+1} = x^2\left(\frac{1}{1-(-x^2)}\right) = x^2\left(1 - x^2 + x^4 - x^6 + \cdots\right) = x^2 - x^4 + x^6 - x^8 + \cdots$$

2.(c) What is the sum of the power series $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{(2x)^k}{k!}$?

Solution: use $e^z = \displaystyle\sum_{k=0}^{\infty}\frac{z^k}{k!}$, for all z. So

$$\sum_{k=0}^{\infty}(-1)^k \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-2x)^k}{k!} = e^{-2x}.$$

3. Calculate $\dfrac{d^2y}{dx^2}$ at the point $(x, y) = (2, 2)$ for each of the two curves:

(a) the parametric curve with $x = 1 + t^3$, $y = t + t^2$.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+2t}{3t^2} = \frac{1}{3t^2} + \frac{2}{3t};$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{-\frac{2}{3t^3} - \frac{2}{3t^2}}{3t^2}.$$

At $(x, y) = (2, 2), t = 1$ so

$$\frac{d^2y}{dx^2} = \frac{-\frac{2}{3} - \frac{2}{3}}{3} = -\frac{4}{9}.$$

(b) the polar curve with $r = \sqrt{8}\, e^{\theta - \pi/4}$.

Solution: the parametric equations of the spiral are

$$x = r\cos\theta = \sqrt{8}\, e^{\theta-\pi/4}\cos\theta, \quad y = r\sin\theta = \sqrt{8}\, e^{\theta-\pi/4}\sin\theta.$$

So

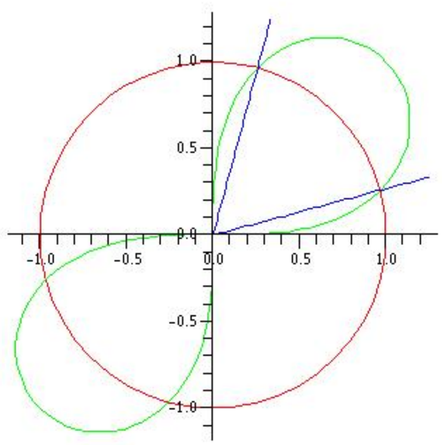
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sqrt{8}\, e^{\theta-\pi/4}\sin\theta + \sqrt{8}\, e^{\theta-\pi/4}\cos\theta}{\sqrt{8}\, e^{\theta-\pi/4}\cos\theta - \sqrt{8}\, e^{\theta-\pi/4}\sin\theta} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

At $(x, y) = (2, 2), \theta = \pi/4 \Rightarrow \dfrac{dy}{dx}$ is undefined. Hence $\dfrac{d^2y}{dx^2}$ is also undefined.

4. Find the area of the region that is outside the circle with polar equation $r = 1$ but inside the curve with polar equation $r^2 = 2\,\sin(2\theta)$.

Solution: at the intersection, in the first quadrant,

$$2\,\sin(2\theta) = 1 \Rightarrow \sin(2\theta) = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}.$$



The region in question is outside the red circle and inside the green curve. Its area is given by

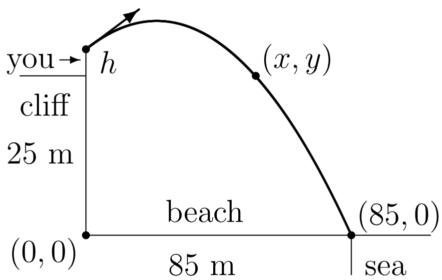
$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_{\pi/12}^{5\pi/12} (2\sin(2\theta) - 1)\; d\theta\right) \\ &= [-\cos(2\theta) - \theta]_{\pi/12}^{5\pi/12} \\ &= -\cos(5\pi/6) - 5\pi/12 + \cos(\pi/6) + \pi/12 \\ &= \sqrt{3} - \frac{\pi}{3} \end{aligned}$$

Note: The green curve with polar equation $r^2 = 2\sin(2\theta)$ has *two* loops, since $r^2 \geq 0$ requires that $\sin(2\theta) \geq 0$. Now

$$\sin(2\theta) \geq 0 \Leftrightarrow 0 \leq \theta \leq \frac{\pi}{2} \text{ or } \pi \leq \theta \leq \frac{3\pi}{2}.$$

So for $r = \sqrt{2\sin(2\theta)}$ there is one loop in the first quadrant and one loop in the third quadrant. For $r = -\sqrt{2\sin(2\theta)} < 0$ the two loops just retrace the two loops with $r > 0$.

5. You are standing on top of a cliff 25 m above a beach which is 85 m from the sea shore. If you can throw a stone at a maximum speed of 25 m/sec, at least how tall must you be to be able to throw a stone from the top of the cliff into the sea?



For simplicity of calculation, assume the acceleration due to gravity is 10 m/sec^2, and ignore air resistance. Use the diagram at the left to set up your solution. Setting up the problem will be worth half the marks.

Solution: Let your height be h; let $x_0 = 0, y_0 = 25 + h, v_0 = 25$ and let α be the angle to the horizontal at which you throw the stone. Then the trajectory of the stone is described parametrically by

$$x = 25\,t\,\cos\alpha, \quad y = 25 + h + 25\,t\,\sin\alpha - 5\,t^2.$$

You will reach the sea if the trajectory passes through the point $(x, y) = (85, 0)$.

$$\begin{aligned} x = 85 \Rightarrow t &= 3.4\sec\alpha \\ \Rightarrow y &= 25 + h + 25(3.4\sec\alpha)\sin\alpha - 5(3.4\sec\alpha)^2 \\ &= 25 + h + 85\tan\alpha - 57.8(1 + \tan^2\alpha) \\ &= -32.8 + h + 85\tan\alpha - 57.8\tan^2\alpha. \\ \text{So } y = 0 &\Rightarrow 57.8\tan^2\alpha - 85\tan\alpha + 32.8 - h = 0. \end{aligned}$$

Calculation 1:

$$\tan\alpha = \frac{85 \pm \sqrt{85^2 - 4(57.8)(32.8) + 4(57.8)h}}{115.6} = \frac{85 \pm \sqrt{231.2h - 358.36}}{115.6}$$

This will be physically possible if and only if $231.2h - 358.36 \geq 0 \Leftrightarrow h \geq 1.55$.

Calculation 2:

$$h = 57.8\tan^2\alpha - 85\tan\alpha + 32.8 \Rightarrow \frac{dh}{d\alpha} = 115.6\tan\alpha\,\sec^2\alpha - 85\sec^2\alpha.$$

So $\dfrac{dh}{d\alpha} = 0 \Leftrightarrow \tan\alpha = \dfrac{85}{115.6} = \dfrac{25}{34}$, and the minimum value of h is

$$h = 57.8\left(\frac{25}{34}\right)^2 - 85\left(\frac{25}{34}\right) + 32.8 = 1.55.$$

(NB: $\tan\alpha = \dfrac{25}{34} \Rightarrow \alpha \simeq 36.33°$, so it is an oversimplification to let $\alpha = 0°$ or $45°$)

Conclusion: Based on either calculation, you have to be at least 1.55 m tall to be able to throw a stone from the cliff to the sea.

Other solutions to this question can be found on the page of Alternate Calculations at the end of this exam.

6. Find and classify all the critical points of the function $f(x, y) = 4xy - x^4 - y^4$.

Solution: Let $z = f(x, y)$. Then

$$\frac{\partial z}{\partial x} = 4y - 4x^3 \text{ and } \frac{\partial z}{\partial y} = 4x - 4y^3.$$

Critical points:

$$\left.\begin{array}{rcl} 4y - 4x^3 & = & 0 \\ 4x - 4y^3 & = & 0 \end{array}\right\} \Rightarrow \left.\begin{array}{rcl} y & = & x^3 \\ x & = & y^3 \end{array}\right\} \Rightarrow y = x^3 \text{ and } x = \left(x^3\right)^3 = x^9.$$

Now

$$x = x^9 \Rightarrow x = 0 \text{ or } x^8 = 1 \Leftrightarrow (x^2 - 1)(x^2 + 1)(x^4 + 1) = 0.$$

Thus $x = 0$ or $x = \pm 1$. So there are three critical points of f:

$$(x, y) = (0, 0), (1, 1) \text{ and } (-1, -1).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = -12x^2, \ \frac{\partial^2 z}{\partial y^2} = -12y^2, \ \frac{\partial^2 z}{\partial x \partial y} = 4$$

and

$$\Delta = 144x^2y^2 - 16.$$

At $(x, y) = (0, 0)$,

$$\Delta = -16 < 0,$$

so f has a saddle point at $(x, y, z) = (0, 0, 0)$.

At $(x, y) = \pm(1, 1)$,

$$\Delta = 128 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = -12 < 0,$$

so f has a maximum value (of $z = 2$) at both $(x, y) = (1, 1)$ and $(-1, -1)$.

7. Approximate

$$\int_0^{\pi/6} \frac{\cos x\, dx}{(1+\sin^4 x)^{2/3}}$$

to within 10^{-5}, and explain why your approximation is correct to within 10^{-5}.

Solution: First, let $u = \sin x$. Then

$$\int_0^{\pi/6} \frac{\cos x\, dx}{(1+\sin^4 x)^{2/3}} = \int_0^{1/2} \frac{du}{(1+u^4)^{2/3}} = \int_0^{1/2} (1+u^4)^{-2/3}\, du$$

Now use the binomial series

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \cdots$$

with $z = u^4$ and $\alpha = -2/3$. Then

$$\begin{aligned}
&\int_0^{0.5} (1+u^4)^{-2/3}\, du \\
= \;& \int_0^{0.5} \left(1 - \frac{2}{3}u^4 + \frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2!}(u^4)^2 + \frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{3!}(u^4)^3 + \cdots\right) du \\
= \;& \int_0^{0.5} \left(1 - \frac{2}{3}u^4 + \frac{5}{9}u^8 - \frac{40}{81}u^{12} + \cdots\right) du \\
= \;& \left[u - \frac{2}{15}u^5 + \frac{5}{81}u^9 - \frac{40}{1053}u^{13} + \cdots\right]_0^{0.5} \\
= \;& 0.5 - 0.004166666\ldots + 0.000120563\ldots - 4.637 \times 10^{-6} + \cdots \\
= \;& 0.495953897\ldots
\end{aligned}$$

which is correct to within $4.637 \times 10^{-6} < 10^{-5}$, by the alternating series remainder term. You *must* refer to the alternating series remainder term to get full marks.

Note: see the Alternate Calculations at the end of these solutions to see what the solution looks like if you don't first make the substitution $u = \sin x$.

8. The trajectory of a particle in space is given by $\mathbf{r} = (t-5)\,\mathbf{i} + t^2\,\mathbf{j} + 2\sqrt{t}\,\mathbf{k}$. Find the point on the trajectory that is closest to the origin, $(0,0,0)$.

Solution: minimize the distance from the origin to the curve, $\|\mathbf{r}\|$, or more conveniently, $\|\mathbf{r}\|^2 = (t-5)^2 + t^4 + 4t$.

$$\frac{d\,\|\mathbf{r}\|^2}{dt} = 2(t-5) + 4t^3 + 4 = 4t^3 + 2t - 6 = 2(t-1)(2t^2+2t+3),$$

so the only real critical point is $t = 1$, since $2t^2+2t+3 = 0$ has only complex roots. Now

$$\frac{d^2\|\mathbf{r}\|^2}{dt^2} = 12t^2 + 2 > 0,$$

for all t. So there is a minimum at $t = 1$. Hence the point closest to the origin is $\mathbf{r} = -4\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$; that is, $(x,y,z) = (-4,1,2)$.

Alternate Method:

$$\begin{aligned}
\frac{d\|\mathbf{r}\|^2}{dt} = 0 \quad &\Leftrightarrow \quad 2\frac{d\,\mathbf{r}\cdot\mathbf{r}}{dt} = 0 \\
&\Leftrightarrow \quad 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0 \\
&\Leftrightarrow \quad \mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0 \\
&\Leftrightarrow \quad ((t-5)\,\mathbf{i} + t^2\,\mathbf{j} + 2\sqrt{t}\,\mathbf{k})\cdot\left(\mathbf{i} + 2t\,\mathbf{j} + \frac{1}{\sqrt{t}}\,\mathbf{k}\right) = 0 \\
&\Leftrightarrow \quad t - 5 + 2t^3 + 2 = 0 \\
&\Leftrightarrow \quad 2t^3 + t - 3 = 0 \\
&\Leftrightarrow \quad (t-1)(2t^2+2t+3) = 0 \\
&\Rightarrow \quad t = 1, \text{ since other solutions are complex}
\end{aligned}$$

9. Consider the cycloid $\mathbf{r} = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$, for $0 \le t \le 2\pi$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = (1 - \cos t)\,\mathbf{i} + \sin t\,\mathbf{j};$$

$$\begin{aligned}\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{(1-\cos t)^2 + (\sin t)^2} \\ &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2 - 2\cos t}\end{aligned}$$

(b) [6 marks] Find an arc length parameterization of the cycloid, with reference point $(0, 0)$, for which $t = 0$.

Solution: using an appropriate double angle formula.

$$\begin{aligned} s &= \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \int_0^t \sqrt{2 - 2\cos u}\, du = \int_0^t \sqrt{4\sin^2(u/2)}\, du \\ &= \int_0^t 2\sin(u/2)\, du, \text{ since } 0 \le t \le 2\pi \Rightarrow 0 \le u/2 \le \pi \\ &= [-4\cos(u/2)]_0^t = -4\cos(t/2) + 4 \\ \Rightarrow t &= 2\cos^{-1}\frac{4-s}{4}, \text{ for } 0 \le s \le 8. \end{aligned}$$

So an arc length parameterization of the cycloid is

$$\mathbf{r} = \left(2\cos^{-1}\frac{4-s}{4} - \sin\left(2\cos^{-1}\frac{4-s}{4}\right)\right)\mathbf{i} + \left(1 - \cos\left(2\cos^{-1}\frac{4-s}{4}\right)\right)\mathbf{j}.$$

Alternate Method of Integration: this is trickier!

$$s = \int_0^t \sqrt{2 - 2\cos u}\, du = \sqrt{2}\int_0^t \sqrt{\frac{(1-\cos u)(1+\cos u)}{1+\cos u}}\, du = \sqrt{2}\int_0^t \sqrt{\frac{\sin^2 u}{1+\cos u}}\, du$$

$$= \begin{cases} \sqrt{2}\displaystyle\int_0^t \frac{\sin u\, du}{\sqrt{1+\cos u}}, & \text{if } 0 \le t \le \pi \\ 4 - \sqrt{2}\displaystyle\int_\pi^t \frac{\sin u\, du}{\sqrt{1+\cos u}}, & \text{if } \pi \le t \le 2\pi \end{cases} = \begin{cases} 4 - 2\sqrt{2}\sqrt{1+\cos t}, & \text{if } 0 \le t \le \pi \\ 4 + 2\sqrt{2}\sqrt{1+\cos t}, & \text{if } \pi \le t \le 2\pi \end{cases}$$

$$\Rightarrow s - 4 = \pm 2\sqrt{2}\sqrt{1+\cos t} \Rightarrow (s-4)^2 = 8(1+\cos t) \Rightarrow t = \cos^{-1}\left(\frac{s^2 - 8s + 8}{8}\right)$$

Alternate Calculations:

5. If

$$\tan\alpha = \frac{85 \pm \sqrt{231.2\,h - 358.36}}{115.6} \text{ and } h = 1.55,$$

then $\tan\alpha = 85/115.6$ and $\alpha \simeq 36.33°$. So it is an oversimplification to assume that either $\alpha = 0°$ or $\alpha = 45°$. However, both of these cases are much easier to solve:

$$\alpha = 0° \Rightarrow (x, y) = (25t, 25 + h - 5t^2) = (85, 0) \Rightarrow (t, h) = (3.4, 32.8)$$

and

$$\begin{aligned} \alpha = 45° \quad &\Rightarrow \quad (x, y) = (25\,t/\sqrt{2}, 25 + h + 25\,t/\sqrt{2} - 5t^2) = (85, 0) \\ &\Rightarrow \quad (t, h) = (3.4\sqrt{2}, 5.6) \end{aligned}$$

Another **correct** (but messier) way to solve Question 5 is to solve $y = 0$ for $t > 0$:

$$\begin{aligned} & 25 + h + 25\,t\,\sin\alpha - 5\,t^2 = 0 \\ \Rightarrow \quad & t = \frac{5}{2}\sin\alpha + \frac{1}{10}\sqrt{625\sin^2\alpha + 500 + 20h} \\ \Rightarrow \quad & x = 25\cos\alpha\left(\frac{5}{2}\sin\alpha + \frac{1}{10}\sqrt{625\sin^2\alpha + 500 + 20h}\right) = 85 \\ \Rightarrow \quad & 34\sec\alpha - 25\sin\alpha = \sqrt{625\sin^2\alpha + 500 + 20h} \\ \Rightarrow \quad & 1156\sec^2\alpha - 1700\tan\alpha + 625\sin^2\alpha = 625\sin^2\alpha + 500 + 20h \\ \Rightarrow \quad & 1156\tan^2\alpha - 1700\tan\alpha + 656 - 20h = 0 \\ \Rightarrow \quad & 57.8\tan^2\alpha - 85\tan\alpha + 32.8 - h = 0, \end{aligned}$$

which is the same quadratic for $\tan\alpha$ as in the first solution to #5.

Still **another** (even messier) way to solve Question 5 is to find the angle that maximizes x. Using x in terms of α, as above: $dx/d\alpha = 0 \Rightarrow$

$$\sin\alpha\left(\frac{5}{2}\sin\alpha + \frac{1}{10}\sqrt{625\sin^2\alpha + 500 + 20h}\right) = \cos\alpha\left(5\cos\alpha + \frac{125\sin\alpha\cos\alpha}{\sqrt{625\sin^2\alpha + 500 + 20h}}\right)$$

$$\Rightarrow \sin\alpha = \frac{25}{\sqrt{1750 + 20h}},$$

after much work! Then

$$\cos\alpha = \sqrt{\frac{1125 + 20h}{1750 + 20h}}$$

and

$$x = 25\sqrt{\frac{1125 + 20h}{1750 + 20h}}\left(\frac{5}{2}\frac{25}{\sqrt{1750 + 20h}} + \frac{1}{10}\sqrt{\frac{625^2}{1750 + 20h} + 500 + 20h}\right),$$

whence $x = 85 \Rightarrow h = 1.55$, as before, but it is tedious! (Compare: $h = 1.55 \Rightarrow \sin\alpha = 25/\sqrt{1781} \Rightarrow \alpha \simeq 36.33°$, as before.)

7. If you don't simplify the integral(s) with a substitution, things look like:

$$
\begin{aligned}
&\int_0^{\pi/6} \frac{\cos x\,dx}{(1+\sin^4 x)^{2/3}} \\
=\ &\int_0^{\pi/6} \cos x\,(1+\sin^4 x)^{-2/3}\,dx \\
=\ &\int_0^{\pi/6} \cos x\,\left(1-\frac{2}{3}\sin^4 x+\frac{5}{9}\sin^8 x-\frac{40}{81}\sin^{12} x+\cdots\right)\,dx \\
=\ &\int_0^{\pi/6} \cos x\,dx-\frac{2}{3}\int_0^{\pi/2} \sin^4 x\,\cos x\,dx+\frac{5}{9}\int_0^{\pi/2} \sin^8 x\,\cos x\,dx-\frac{40}{81}\int_0^{\pi/2} \sin^{12} x\,\cos x\,dx+\cdots \\
=\ &[\sin x]_0^{\pi/6}-\frac{2}{3}\left[\frac{\sin^5 x}{5}\right]_0^{\pi/6}+\frac{5}{9}\left[\frac{\sin^9 x}{9}\right]_0^{\pi/6}-\frac{40}{81}\left[\frac{\sin^{13} x}{13}\right]_0^{\pi/6}+\cdots \\
=\ &0.5-\frac{2}{15}(0.5)^5+\frac{5}{81}(0.5)^9-\frac{40}{1053}(0.5)^{13}+\cdots
\end{aligned}
$$

which is the same series as in the solution presented previously.

9. The parameterization given in the solutions, for $0 \le s \le 8$, can be simplified using trigonometry:

$$
\begin{aligned}
&\left(2\,\cos^{-1}\frac{4-s}{4}-\sin\left(2\cos^{-1}\frac{4-s}{4}\right)\right)\mathbf{i}+\left(1-\cos\left(2\cos^{-1}\frac{4-s}{4}\right)\right)\mathbf{j} \\
=\ &\left(2\,\cos^{-1}\frac{4-s}{4}-2\left(\sin\left(\cos^{-1}\frac{4-s}{4}\right)\cos\left(\cos^{-1}\frac{4-s}{4}\right)\right)\right)\mathbf{i}+\left(2\sin^2\left(\cos^{-1}\frac{4-s}{4}\right)\right)\mathbf{j} \\
=\ &\left(2\,\cos^{-1}\frac{4-s}{4}-\frac{4-s}{2}\left(\frac{\sqrt{8s-s^2}}{4}\right)\right)\mathbf{i}+\frac{8s-s^2}{8}\mathbf{j} \\
=\ &\left(2\,\cos^{-1}\frac{4-s}{4}-\frac{(4-s)\sqrt{8s-s^2}}{8}\right)\mathbf{i}+\frac{8s-s^2}{8}\mathbf{j}
\end{aligned}
$$

So

$$x = 2\,\cos^{-1}\frac{4-s}{4}-\frac{(4-s)\sqrt{8s-s^2}}{8},\quad y=\frac{8s-s^2}{8}.$$