University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2010**
First Year - CHE, CIV, IND, LME, MEC, MSE

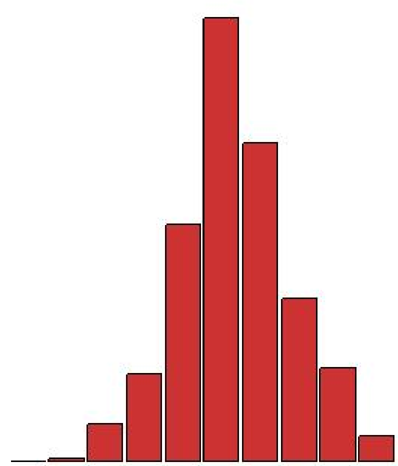
MAT187H1S - CALCULUS II
Exam Type: A

General Comments:

1. The challenge in Question 9 is to set up a combination of integrals that gives the desired area. There is no single integral that solves the problem in one step.
2. In Question 11, many students did not solve the system of equations for the critical points. Instead, many students found one or two solutions by inspection. This is not good enough to get full marks. Also: $(0, 0)$ is *not* a critical point; it's not even in the domain of f.
3. The three parts of Question 13 are not all best solved in the same way!
4. There is a section, Alternate Calculations and More Details, added at the end of these solutions for your interest.

Breakdown of Results: 451 students wrote this exam. The marks ranged from 19% to 100%, and the average mark was only 57.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	1.8%
A	8.5%	80-89%	6.7%
B	11.5%	70-79%	11.5%
C	22.6%	60-69%	22.6%
D	31.5%	50-59%	31.5%
F	25.9%	40-49%	16.8%
		30-39%	6.2%
		20-29%	2.7%
		10-19%	0.2%
		0-9%	0.0%



1. The solution to the initial value problem $\dfrac{dy}{dx} = \dfrac{2x}{3y^2}$, $y(0) = 5$ is

(a) $y = x^2 + 5$

(b) $y = x^{2/3} + 5$

(c) $y = \sqrt[3]{x^2 + 5^3}$

(d) $y = (x + 5)^{2/3}$

Solution: separate variables.

$$\int 3y^2\,dy = \int 2x\,dx \Leftrightarrow y^3 = x^2 + C.$$

$$(x, y) = (0, 5) \Rightarrow C = 5^3.$$

The answer is (c)

2. The general solution to the differential equation $\dfrac{dy}{dx} - \dfrac{y}{x} = 1$, for $x > 0$, is

(a) $y = x \ln x + Cx$

(b) $y = x \ln x + C$

(c) $y = x^2 + Cx$

(d) $y = \dfrac{\ln x}{x} + \dfrac{C}{x}$

Solution:

$$\mu = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x};$$

$$\begin{aligned} y &= \frac{1}{\mu} \int \mu\,dx \\ &= x\left(\int \frac{1}{x}\,dx\right) \\ &= x\left(\ln x + C\right). \end{aligned}$$

The answer is (a)

3. The general solution to the differential equation $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$ is

(a) $y = C\,e^{-3x}$

(b) $y = A\,e^{-3x} + B\,x\,e^{-3x}$

(c) $y = A\,e^{-3x}\,\sin(3x) + B\,e^{-3x}\,\cos(3x)$

(d) $y = C\,x\,e^{-3x}$

Solution: the only root of the auxiliary quadratic is $r = -3$, repeated. So the solution is

$$y = Ae^{-3x} + Bxe^{-3x}.$$

The answer is (b)

4. What is the interval of convergence of the power series $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k (x+5)^k$?

(a) $\left(-\frac{3}{4}, \frac{3}{4}\right)$

(b) $\left(-\frac{15}{4}, \frac{15}{4}\right)$

(c) $\left(\frac{11}{3}, \frac{19}{3}\right)$

(d) $\left(-\frac{19}{3}, -\frac{11}{3}\right)$

Solution: this is an infinite geometric series and converges if and only if

$$\left|\frac{3(x+5)}{4}\right| < 1 \Leftrightarrow |x+5| < \frac{4}{3} \Leftrightarrow -\frac{4}{3} < x+5 < \frac{4}{3}$$

The answer is (d)

5. The first four non-zero terms of the Maclaurin series for $\tan^{-1}\left(\frac{x^2}{2}\right)$ are:

(a) $\frac{x}{2} + \frac{x^3}{24} + \frac{x^5}{160} + \frac{x^7}{896}$

(b) $\frac{x}{2} - \frac{x^3}{24} + \frac{x^6}{160} - \frac{x^7}{896}$

(c) $\frac{x^2}{2} - \frac{x^6}{24} + \frac{x^{10}}{160} - \frac{x^{14}}{896}$

(d) $\frac{x^2}{2} + \frac{x^6}{24} + \frac{x^{10}}{160} + \frac{x^{14}}{896}$

Solution: use

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

and let $z = \frac{x^2}{2}$.

The answer is (c)

6. The infinite series $\sum_{k=0}^{\infty} (-1)^k \frac{k^2}{\sqrt{k^6+1}}$

(a) converges absolutely.

(b) converges conditionally.

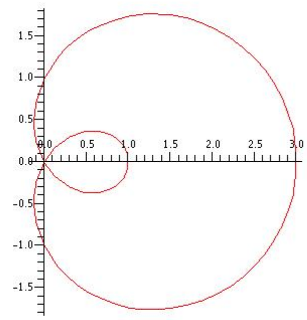
(c) diverges.

(d) is called the harmonic series.

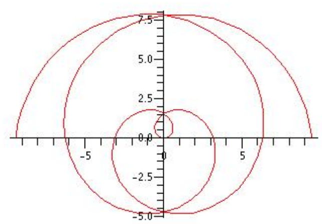
Solution: $\sum_{k=0}^{\infty} (-1)^k \frac{k^2}{\sqrt{k^6+1}}$ converges by the alternating series test. But $\sum_{k=0}^{\infty} \frac{k^2}{\sqrt{k^6+1}}$ diverges by the limit comparison test.

The answer is (b)

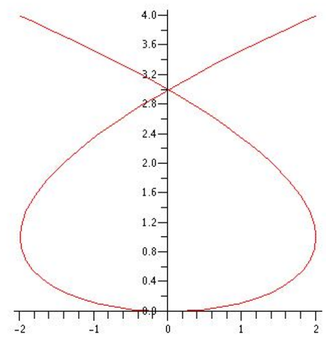
7. [2 marks for each graph.] Match the graphs A through F with their correct vector or polar equations, by putting the letter of the graph in the blank line beside the single appropriate vector or polar equation at the bottom of this page.



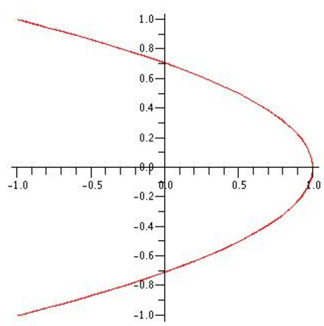
A



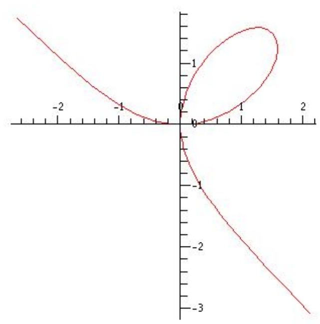
B



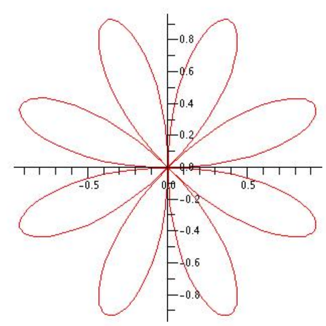
C



D



E



F

___ $r = 1 + 2\sin\theta$	___ $r = 1 - 2\cos\theta$	A $r = 1 + 2\cos\theta$
___ $r = \cos 4\theta$	F $r = \sin 4\theta$	___ $r = \cos 8\theta$
___ $r = \sin 8\theta$	B $r = \theta$	___ $r = 1 - 2\sin\theta$
D $\mathbf{r} = \cos 2t\,\mathbf{i} + \sin t\,\mathbf{j}$	___ $\mathbf{r} = \sin 2t\,\mathbf{i} + \cos t\,\mathbf{j}$	C $\mathbf{r} = (t^3 - 3t)\,\mathbf{i} + t^2\,\mathbf{j}$
___ $\mathbf{r} = t^2\,\mathbf{i} + (t^3 - 3t)\,\mathbf{j}$	E $\mathbf{r} = \dfrac{3t}{1+t^3}\,\mathbf{i} + \dfrac{3t^2}{1+t^3}\,\mathbf{j}$	___ $\mathbf{r} = \dfrac{t^3}{1+t^2}\,\mathbf{i} + \dfrac{2t^2}{1+t^2}\,\mathbf{j}$

8. [12 marks] **Given** that the acceleration of a particle at time t is $\mathbf{a} = 2\,\mathbf{i} - 9.8\,\mathbf{k}$, its velocity at $t = 0$ is $\mathbf{v}_0 = 30\,\mathbf{j} + 30\,\mathbf{k}$, and its position at $t = 0$ is $\mathbf{r}_0 = 10\,\mathbf{k}$, **find** the position and the speed of the particle when $z = 0$, for $t > 0$. (Use your calculator and approximate your answers to one decimal place.)

Solution:

$$\mathbf{v} = \int \mathbf{a}\,dt = \int (2\,\mathbf{i} - 9.8\,\mathbf{k})\;dt = 2t\,\mathbf{i} - 9.8t\,\mathbf{k} + \mathbf{C}.$$

Use the initial condition to find that

$$\mathbf{C} = \mathbf{v}_0 = 30\,\mathbf{j} + 30\,\mathbf{k}.$$

Thus

$$\mathbf{v} = 2t\,\mathbf{i} + 30\,\mathbf{j} + (30 - 9.8t)\,\mathbf{k}.$$

Then

$$\mathbf{r} = \int \mathbf{v}\,dt = \int (2t\,\mathbf{i} + 30\,\mathbf{j} + (30 - 9.8t)\,\mathbf{k})\;dt = t^2\,\mathbf{i} + 30t\,\mathbf{j} + (30t - 4.9t^2)\,\mathbf{k} + \mathbf{D}.$$

Use the initial condition to find that

$$\mathbf{D} = \mathbf{r}_0 = 10\,\mathbf{k}.$$

Thus

$$\mathbf{r} = t^2\,\mathbf{i} + 30t\,\mathbf{j} + (10 + 30t - 4.9t^2)\,\mathbf{k}.$$

Now

$$\begin{aligned} z = 0 \quad &\Rightarrow \quad 10 + 30t - 4.9t^2 = 0 \\ &\Rightarrow \quad t = \frac{30 \pm \sqrt{900 + 40(4.9)}}{9.8} \\ &\Rightarrow \quad t \simeq 6.4,\ \text{since } t > 0 \end{aligned}$$

At $t = 6.4$, the position of the particle is

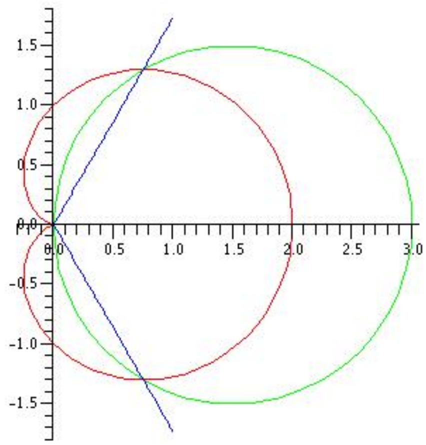
$$x \simeq 41.0 \text{ and } y \simeq 192.0$$

and its speed is

$$\|12.8\,\mathbf{i} + 30\,\mathbf{j} - 32.72\,\mathbf{k}\| = \sqrt{(12.8)^2 + 30^2 + (-32.72)^2} \simeq 46.2$$

9. [12 marks] Find the area of the region that is inside both the circle with polar equation $r = 3\cos\theta$ and the cardioid with polar equation $r = 1 + \cos\theta$.

Comments: at the intersection, $3\cos\theta = 1 + \cos\theta \Leftrightarrow \cos\theta = \dfrac{1}{2} \Leftrightarrow \theta = \pm\dfrac{\pi}{3}$.



The region in question is inside the red cardioid and inside the green circle. The area of this region can't be calculated in one step. The integral

$$A_1 = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((3\cos\theta)^2 - (1+\cos\theta)^2\right)\, d\theta$$

gives the area of crescent shaped region inside the green circle but outside the red cardioid.

The integral

$$A_2 = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (1+\cos\theta)^2\, d\theta$$

gives the area of the region inside the red cardioid between the two blue rays $\theta = \pm\dfrac{\pi}{3}$, while the integral

$$A_3 = \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2\, d\theta$$

gives the area of the sliver in the first quadrant inside the green circle but above the blue ray $\theta = \pi/3$. Finally, the integral

$$A_4 = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (3\cos\theta)^2\, d\theta$$

gives the area within the green circle, which has radius $a = \dfrac{3}{2}$, so $A_4 = \dfrac{9}{4}\pi$.

Solution: let A be the area of the region that is inside both the red cardioid and the green circle. Use either $A = A_4 - A_1$ or $A = A_2 + 2A_3$. Using the former:

$$\begin{aligned} A &= \frac{9}{4}\pi - \int_0^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right)\, d\theta = \frac{9}{4}\pi - \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta \\ &= \frac{9}{4}\pi - [3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\pi/3} = \frac{9}{4}\pi - \pi = \frac{5}{4}\pi. \end{aligned}$$

10. [10 marks] Approximate

$$\int_0^{0.5} \frac{dx}{(1+x^4)^{1/3}}$$

to within 10^{-5}, and explain why your approximation is correct to within 10^{-5}.

Solution: use the binomial series

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots$$

with $z = x^4$ and $\alpha = -1/3$. Then

$$\begin{aligned}
&\int_0^{0.5} (1+x^4)^{-1/3}\,dx \\
= &\int_0^{0.5} \left(1 - \frac{1}{3}x^4 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}(x^4)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}(x^4)^3 + \cdots\right) dx \\
= &\int_0^{0.5} \left(1 - \frac{1}{3}x^4 + \frac{2}{9}x^8 - \frac{14}{81}x^{12} + \cdots\right) dx \\
= &\left[x - \frac{1}{15}x^5 + \frac{2}{81}x^9 - \frac{14}{1053}x^{13} + \cdots\right]_0^{0.5} \\
= &\ 0.5 - 0.002083333333\ldots + 0.00004822530864\ldots - 1.622967118 \times 10^{-6} + \cdots \\
= &\ 0.497964892\ldots
\end{aligned}$$

which is correct to within $1.62296718 \times 10^{-6} < 10^{-5}$, by the alternating series remainder term. You *must* refer to the alternating series remainder term to get full marks.

11. [10 marks] Find and classify all the critical points of the function $f(x,y) = x^2 + y^2 + \dfrac{2}{xy}$.

Solution: Let $z = f(x,y)$, for $xy \neq 0$. Then

$$\frac{\partial z}{\partial x} = 2x - \frac{2}{x^2 y} \text{ and } \frac{\partial z}{\partial y} = 2y - \frac{2}{xy^2}.$$

Critical points:

$$\left.\begin{array}{rcl} 2x - \dfrac{2}{x^2y} & = & 0 \\ 2y - \dfrac{2}{xy^2} & = & 0 \end{array}\right\} \Rightarrow \left.\begin{array}{rcl} x^3y & = & 1 \\ xy^3 & = & 1 \end{array}\right\} \Rightarrow \frac{x^2}{y^2} = 1 \text{ and } y = \frac{1}{x^3}.$$

Case 1: If $y = x$, then

$$x = \frac{1}{x^3} \Leftrightarrow x^4 = 1 \Rightarrow x = \pm 1.$$

So two critical points are

$$(x,y) = (1,1) \text{ and } (x,y) = (-1,-1).$$

Case 2: If $y = -x$, then

$$-x = \frac{1}{x^3} \Leftrightarrow x^4 = -1 \Rightarrow x \notin \mathbb{R}$$

so there are no critical points in this case.

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 2 + \frac{4}{x^3 y},\ \frac{\partial^2 z}{\partial y^2} = 2 + \frac{4}{xy^3},\ \frac{\partial^2 z}{\partial x \partial y} = \frac{2}{x^2y^2}$$

and

$$\Delta = \left(2 + \frac{4}{x^3y}\right)\left(2 + \frac{4}{xy^3}\right) - \left(\frac{2}{x^2y^2}\right)^2.$$

At both critical points,

$$\Delta = 32 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = 6 > 0,$$

so f has a minimum value (of $z = 4$) at both critical points $(x,y) = (1,1)$ and $(-1,-1)$.

12. [10 marks] Consider the curve $\mathbf{r} = \sin e^t\,\mathbf{i} + \cos e^t\,\mathbf{j} + \sqrt{3}\,e^t\,\mathbf{k}$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = e^t\cos e^t\,\mathbf{i} - e^t\sin e^t\,\mathbf{j} + \sqrt{3}\,e^t\,\mathbf{k};$$

$$\begin{aligned}
\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{(e^t\cos e^t)^2 + (-e^t\sin e^t)^2 + (\sqrt{3}\,e^t)^2} \\
&= \sqrt{e^{2t}\cos^2 e^t + e^{2t}\sin^2 e^t + 3\,e^{2t}} \\
&= \sqrt{4\,e^{2t}} \\
&= 2\,e^t
\end{aligned}$$

(b) [4 marks] Find an arc length parametrization of the above curve, with reference point $(0, -1, \sqrt{3}\,\pi)$, for which $t = \ln\pi$.

Solution:

$$s = \int_{\ln\pi}^{t} \left\|\frac{d\mathbf{r}}{du}\right\| du = [2\,e^u]_{\ln\pi}^{t} = 2\,e^t - 2\pi \Leftrightarrow e^t = \frac{s+2\pi}{2}$$

So an arc length parametrization of the curve is

$$\mathbf{r} = \sin\left(\frac{s+2\pi}{2}\right)\mathbf{i} + \cos\left(\frac{s+2\pi}{2}\right)\mathbf{j} + \sqrt{3}\left(\frac{s+2\pi}{2}\right)\mathbf{k}.$$

Alternate Answer: the above parametrization can be simplified, using trigonometric identities. To wit:

$$\begin{aligned}
\mathbf{r} &= \sin\left(\frac{s+2\pi}{2}\right)\mathbf{i} + \cos\left(\frac{s+2\pi}{2}\right)\mathbf{j} + \sqrt{3}\left(\frac{s+2\pi}{2}\right)\mathbf{k} \\
&= \sin\left(\frac{s}{2}+\pi\right)\mathbf{i} + \cos\left(\frac{s}{2}+\pi\right)\mathbf{j} + \sqrt{3}\left(\frac{s}{2}+\pi\right)\mathbf{k} \\
&= -\sin\left(\frac{s}{2}\right)\mathbf{i} - \cos\left(\frac{s}{2}\right)\mathbf{j} + \sqrt{3}\left(\frac{s}{2}+\pi\right)\mathbf{k}
\end{aligned}$$

13. [10 marks] The point $(x, y) = (2, 4)$ is a critical point on the graph of each of the following curves. (You don't have to check this!) Determine in each case if the critical point is a relative maximum point, a relative minimum point, or neither.

(a) [2 marks] The graph of the function defined by the power series

$$y = 4 + \frac{1}{4}(x-2)^2 - \frac{1}{24}(x-2)^3 + \frac{1}{192}(x-2)^4 - \frac{1}{1920}(x-2)^5 + \cdots$$

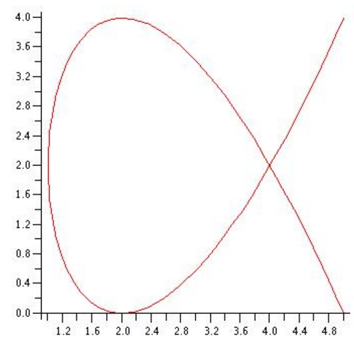
Solution: $\dfrac{y''(2)}{2!} = \dfrac{1}{4} \Rightarrow y''(2) = \dfrac{1}{2} > 0$, so $y = 4$ is a relative minimum value.

(b) [4 marks] The curve with parametric equations $x = 1 + t^2$, $y = 2 + 3t - t^3$.

Solution: observe $(x, y) = (2, 4) \Leftrightarrow t = 1$.

$$\frac{dy}{dx} = \frac{3 - 3t^2}{2t} = \frac{3}{2t} - \frac{3t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{-\dfrac{3}{2t^2} - \dfrac{3}{2}}{2t} = -\frac{3}{4t}\left(\frac{1}{t^2} + 1\right)$$



At $t = 1$, $\dfrac{dy}{dx} = 0$, and $\dfrac{d^2y}{dx^2} = -\dfrac{3}{2} < 0$, so the critical point $(x, y) = (2, 4)$ is a relative maximum point. (You do not need to sketch the curve.)

(c) [4 marks] The polar curve with polar equation $r = 3\sin\theta + 4\cos\theta$.

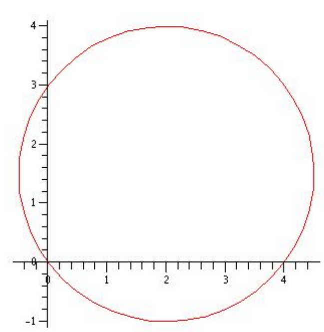
Solution: $r^2 = 3r\sin\theta + 4r\cos\theta \Rightarrow x^2 + y^2 = 3y + 4x$. Now differentiate implicitly:

$$\frac{dy}{dx} = -\frac{2x-4}{2y-3};\ \frac{d^2y}{dx^2} = -\frac{2(2y-3) - (2x-4)(2\,dy/dx)}{(2y-3)^2} = -\frac{50}{(2y-3)^3}.$$

At $(x, y) = (2, 4)$, $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = -\dfrac{2}{5} < 0$, so $(2, 4)$ is a maximum point.

Or: complete the squares to find that the curve is a circle with centre $\left(2, \dfrac{3}{2}\right)$ and radius $a = \dfrac{5}{2}$:

$$(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{25}{4}.$$



The point $(2, 4)$ is the top of the circle, so it is a maximum point.

Alternate Calculations and More Details:

4. For the power series $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k (x+5)^k$, $a_k = \left(\frac{3}{4}\right)^k$ and $c = -5$. Then the radius of convergence is

$$R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}} = \frac{4}{3},$$

and the open interval of convergence is

$$(c-R, c+R) = \left(-5 - \frac{4}{3}, -5 + \frac{4}{3}\right) = \left(-\frac{19}{3}, -\frac{11}{3}\right).$$

At $x = -\frac{19}{3}$ the power series becomes $\sum_{k=0}^{\infty}(-1)^k$, which diverges; and at $x = -\frac{11}{3}$ the power series becomes $\sum_{k=0}^{\infty} 1$, which also diverges. So the interval of convergence of the power series is the same as the open interval of convergence.

7. Proceed by doing the "obvious" ones first, and then by looking for salient features to decide among the rest. For instance: A is a limaçon so must have one of the four equations

$$r = 1 + 2\sin\theta,\ r = 1 - 2\sin\theta,\ r = 1 + 2\cos\theta,\ r = 1 - 2\sin\theta;$$

F is an 8-leaved rose, so must have one of the four equations

$$r = \cos 4\theta,\ r = \sin 4\theta,\ r = \cos 8\theta,\ r = \sin 8\theta.$$

C has $y \geq 0$, so could have equation

$$\mathbf{r} = (t^3 - 3t)\,\mathbf{i} + t^2\,\mathbf{j} \text{ or } \mathbf{r} = \frac{t^3}{1+t^2} + \frac{2t^2}{1+t^2}\,\mathbf{j},$$

but only the former has two y-intercepts. Both

$$\mathbf{r} = \cos 2t\,\mathbf{i} + \sin t\,\mathbf{j} \text{ and } \mathbf{r} = \sin 2t\,\mathbf{i} + \cos t\,\mathbf{j}$$

could correspond to D, since in each case $|x| \leq 1$ and $|y| \leq 1$. But eliminating the parameter in the former gives $x = 1 - 2y^2$, which is a parabola, opening to the left. E has points in all quadrants, except the third. So E must correspond to

$$\mathbf{r} = \frac{3t}{1+t^3}\,\mathbf{i} + \frac{3t^2}{1+t^3}\,\mathbf{j}.$$

Finally, $r = \theta$ is the polar equation of an Archimedean spiral, which corresponds to B, *if* you include $\theta < 0$.

9. The calculations for the four integrals are:

$$\begin{aligned}
A_1 &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((3\cos\theta)^2 - (1+\cos\theta)^2\right)\,d\theta \\
&= \int_0^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right)\,d\theta \\
&= \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\,d\theta \\
&= [3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\pi/3} \\
&= \pi.
\end{aligned}$$

$$\begin{aligned}
A_2 &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (1+\cos\theta)^2\,d\theta \\
&= \int_0^{\pi/3} \left(\cos^2\theta + 2\cos\theta + 1\right)\,d\theta \\
&= \int_0^{\pi/3} \left(\frac{3}{2} + \frac{1}{2}\cos 2\theta + 2\cos\theta\right)\,d\theta \\
&= \left[\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta + 2\sin\theta\right]_0^{\pi/3} \\
&= \frac{\pi}{2} + \frac{1}{4}\frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2} \\
&= \frac{\pi}{2} + \frac{9\sqrt{3}}{8}
\end{aligned}$$

$$\begin{aligned}
A_3 &= \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2\,d\theta \\
&= \frac{1}{2}\int_{\pi/3}^{\pi/2} 9\cos^2\theta\,d\theta \\
&= \frac{9}{4}\int_{\pi/3}^{\pi/2} (1+\cos 2\theta)\,d\theta \\
&= \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2} \\
&= \frac{9}{4}\left(\frac{\pi}{2} - \frac{\pi}{3} + 0 - \frac{1}{2}\frac{\sqrt{3}}{2}\right) \\
&= \frac{3}{8}\pi - \frac{9\sqrt{3}}{16}
\end{aligned}$$

$$\begin{aligned}
A_4 &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (3\cos\theta)^2\,d\theta \\
&= \int_0^{\pi/2} 9\cos^2\theta\,d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} (1+\cos 2\theta)\,d\theta \\
&= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= \frac{9}{4}\pi
\end{aligned}$$

13(c) Good luck if you try calculating the derivatives in terms of the parameter θ! The derivatives, supplied by Maple, with much additional simplification, are:

$$\frac{dy}{dx} = \frac{3\sin 2\theta + 4\cos 2\theta}{3\cos 2\theta - 4\sin 2\theta}$$

and

$$\frac{d^2y}{dx^2} = -\frac{50}{(7\cos^2 2\theta + 24\cos 2\theta\sin 2\theta - 16)(3\cos 2\theta - 4\sin 2\theta)}.$$

At the critical point $(x, y) = (2, 4)$, $\tan\theta = 2 \Leftrightarrow \theta = \tan^{-1} 2$ and

$$\sin 2\theta = \frac{4}{5},\quad \cos 2\theta = -\frac{3}{5},\quad \frac{dy}{dx} = 0 \quad\text{and}\quad \frac{d^2y}{dx^2} = -\frac{2}{5} < 0.$$

As before, the point $(x, y) = (2, 4)$ is a maximum point.