University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2009**
First Year - CHE, CIV, IND, LME, MEC, MMS

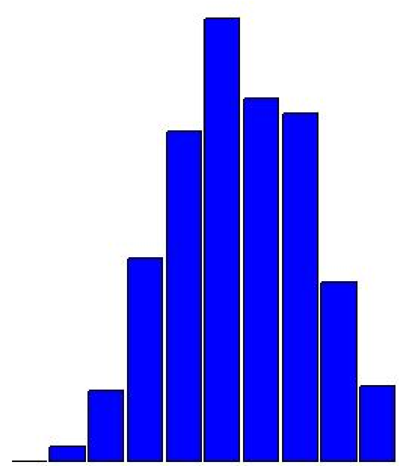
MAT187H1S - CALCULUS II
Exam Type: A

Comments and Alternate Solutions:

1. Questions 11 and 12 were out of the homework. Question 8 was out of the book, but not an assigned homework problem. They are all routine questions.
2. 7(a) converges by the comparison test with $b_n = \dfrac{1}{\sqrt{n}}$ or $\dfrac{1}{n}$, but to get full marks you must show that $a_n > b_n$. It's easier to use the limit comparison test.
3. To get full marks when using the comparison test in 7(b) you have to say $|a_n| < 1/n^2$, since the comparison test assumes both series are positive-term series. You can also use a combination of comparison and integral tests for 7(b).
4. In 7(c), the n-th term goes to infinity, but that is not easy to prove directly. For instance, you can prove that $a_n > n/4$. So 7(c) does diverge by the n-th term test.
5. Question 13 can be done using a straightforward approach; but it is easier to do if you use power series.

Breakdown of Results: 430 students wrote this exam. The marks ranged from 12% to 99%, and the average mark was 58.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.7%
A	12.5%	80-89%	8.8%
B	17.2%	70-79%	17.2%
C	17.9%	60-69%	17.9%
D	21.9%	50-59%	21.9%
F	30.5%	40-49%	16.3%
		30-39%	10%
		20-29%	3.5%
		10-19%	0.7%
		0-9%	0.0%



1. The position $x(t)$ of a particle at time t changes according to the differential equation

$$4x''(t) + 7x'(t) + 3x(t) = 0.$$

The motion of the particle is

(a) simple harmonic motion.

(b) underdamped.

(c) critically damped.

(d) overdamped.

Solution: solve the associated quadratic.

$$4r^2 + 7r + 3 = 0 \Rightarrow r = -\frac{3}{4} \text{ or } -1$$

Both roots are real, so the system is overdamped. The answer is (d).

2. What is the length of the polar curve with polar equation $r = e^{\theta}$, for $0 \le \theta \le 1$?

(a) $e - 1$

(b) $\sqrt{2}$

(c) $\sqrt{2}(e-1)$

(d) $\sqrt{2}e$

Solution:

$$\begin{aligned} \int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^1 \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta \\ &= \sqrt{2}\int_0^1 e^{\theta}\, d\theta \\ &= \sqrt{2}\left[e^{\theta}\right]_0^1 = \sqrt{2}(e-1) \end{aligned}$$

The answer is (c).

3. What is the interval of convergence of the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{4^n \ln n}(x-1)^n$?

(a) $(-3, 5)$

(b) $[-3, 5]$

(c) $[-3, 5)$

(d) $(-3, 5]$

Solution: Check convergence at endpoints. At $x = -3$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{4^n \ln n}(-4)^n = \sum_{n=2}^{\infty} \frac{1}{\ln n},$$

which diverges, by comparison with the harmonic series. At $x = 5$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{4^n \ln n}(4^n) = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n},$$

which converges by the alternating series test.
The answer is (d).

4. How many critical points are there on the curve with parametric equations

$$x = t^3 - 12t,\ \ y = \ln(t^2 + 1),$$

for $t \in \mathbb{R}$?

(a) 1

(b) 2

(c) 3

(d) 4

Solution:

$$\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dt} = \frac{2t}{t^2+1} = 0 \Leftrightarrow t = 0;$$

$$\frac{dy}{dx} \text{ is undefined } \Leftrightarrow \frac{dx}{dt} = 3t^2 - 12 = 0 \Leftrightarrow t = \pm 2.$$

The answer is (c).

5. What is the area bounded by one loop of the curve with polar equation $r^2 = 4\cos\theta$?

(a) 1

(b) 2

(c) 4

(d) 8

Solution: $r^2 = 4\cos\theta = 0 \Rightarrow \theta = \pm\pi/2$.

$$A = 2\left(\frac{1}{2}\int_0^{\pi/2} r^2\, d\theta\right) = \int_0^{\pi/2} 4\cos\theta\, d\theta = [4\sin\theta]_0^{\pi/2} = 4.$$

The answer is (c).

6. What is the arc length of the curve with parametric equations

$$x = t\sin t; y = t\cos t; z = \frac{1}{3}(2t)^{3/2},$$

for $0 \le t \le 2$?

(a) 1

(b) 2

(c) 3

(d) 4

Solution:

$$\begin{aligned}
&\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt \\
= &\int_0^2 \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + (\sqrt{2t})^2}\, dt \\
= &\int_0^2 \sqrt{1 + 2t + t^2}\, dt = \int_0^2 (1+t)\, dt = \left[t + \frac{t^2}{2}\right]_0^2 = 4
\end{aligned}$$

The answer is (d).

7. [12 marks; 4 for each part.] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^{5/2} + 3n^2 - 3n}{n^3 + n^{3/2} + \sqrt{n}}$ ○ Converges ⊗ Diverges

by limit comparison test.

Calculation: use the fact that the p-series with $p = 1/2$ diverges.

$$a_n = \frac{4n^{5/2} + 3n^2 - 3n}{n^3 + n^{3/2} + \sqrt{n}};\, b_n = \frac{1}{\sqrt{n}} \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = 4.$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n^2 + 3n - 6)}{n^2 + 1}$ ⊗ Converges ○ Diverges

by absolute convergence and comparison test.

Calculation: compare with the p-series, $p = 2$, which converges.

$$\sum_{n=0}^{\infty} \left| \frac{\sin(n^2 + 3n - 6)}{n^2 + 1} \right| \le \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} \le \sum_{n=0}^{\infty} \frac{1}{n^2}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(2n)!}$ ○ Converges ⊗ Diverges

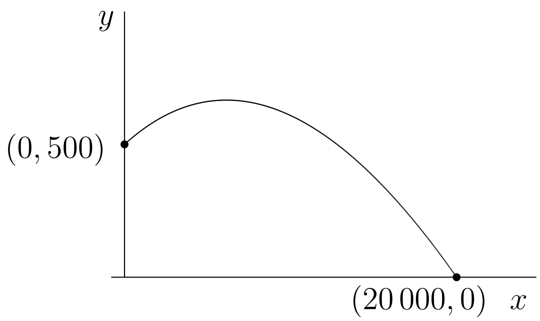
by the ratio test.

Calculation:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^{2n} \cdot \frac{(n+1)^2}{(2n+2)(2n+1)} \right) = \frac{e^2}{4} > 1.$$

8. [12 marks] An artillery gun with muzzle velocity (initial speed) of 1000 ft/sec is located atop a seaside cliff 500 ft high. At what inclination angle(s) should it fire a projectile in order to hit a ship at sea 20,000 ft from the base of the cliff? (Neglect air resistance; use $g = 32$ ft/sec^2.)

Solution: Take $v_0 = 1000$ and let the angle of inclination be α.



Then

$$x = 1000\,t\cos\alpha$$

and

$$y = 500 + 1000\,t\sin\alpha - 16\,t^2.$$

Solve for α and t if $(x, y) = (20\,000, 0)$:

$$1000\,t\cos\alpha = 20\,000 \Rightarrow t = 20\sec\alpha,$$

and

$$\begin{aligned}
& 500 + 1000\,(20\sec\alpha)\sin\alpha - 16\,(400\sec^2\alpha) = 0 \\
\Rightarrow\ & 1 + 40\tan\alpha - 12.8\sec^2\alpha = 0 \\
\Rightarrow\ & 12.8\,(\tan^2\alpha + 1) - 40\tan\alpha - 1 = 0 \\
\Rightarrow\ & 12.8\tan^2\alpha - 40\tan\alpha + 11.8 = 0 \\
\\
\Rightarrow\ & \tan\alpha = \frac{40 \pm \sqrt{40^2 - 4(12.8)(11.8)}}{25.6} \\
\Rightarrow\ & \tan\alpha \simeq 2.795192683 \text{ or } 0.3298073173 \\
\Rightarrow\ & \alpha \simeq 70.3^\circ \text{ or } 18.2^\circ
\end{aligned}$$

So the inclination angle of the artillery gun should be 70.3° or 18.2° from the horizontal.

9. [12 marks] Solve for y as a function of x if $y = 1$ when $x = 0$ and

$$\frac{dy}{dx} + \frac{y}{1+x} = \arctan x.$$

Solution: The integrating factor is

$$\rho = e^{\int \frac{1}{1+x}\,dx} = e^{\ln|1+x|} = |1+x|;$$

take $\rho = (1+x)$ or $-(1+x)$, it makes no difference. Then

$$\begin{aligned} y &= \frac{\int \rho \arctan x \, dx}{\rho} \\ &= \frac{\int (1+x)\arctan x \, dx}{1+x} \end{aligned}$$

Use integration by parts; let $u = \arctan x; dv = (1+x)\,dx$. Then

$$\begin{aligned} \int (1+x)\arctan x \, dx &= uv - \int v\,du \\ &= \frac{1}{2}(1+x)^2 \arctan x - \int \frac{1}{2}(1+x)^2 \frac{1}{1+x^2}\,dx \\ &= \frac{1}{2}(1+x)^2 \arctan x - \frac{1}{2}\int \frac{1+2x+x^2}{1+x^2}\,dx \\ &= \frac{1}{2}(1+x)^2 \arctan x - \frac{1}{2}\int \left(1 + \frac{2x}{1+x^2}\right)dx \\ &= \frac{1}{2}(1+x)^2 \arctan x - \frac{1}{2}\left(x + \ln(1+x^2)\right) + C \end{aligned}$$

Thus the general solution is

$$y = \frac{1}{2}(1+x)\arctan x - \frac{x+\ln(1+x^2)}{2(1+x)} + \frac{C}{1+x}.$$

If $(x, y) = (0, 1)$, then $1 = 0 - 0 + C \Leftrightarrow C = 1$. So the particular soluton is

$$y = \frac{1}{2}(1+x)\arctan x - \frac{x+\ln(1+x^2)}{2(1+x)} + \frac{1}{1+x}.$$

10. [10 marks] Approximate $\int_0^{0.5} (1+x^6)^{3/2}\,dx$ to within 10^{-6}, and explain why your approximation is correct to within 10^{-6}.

Solution: use the binomial series.

$$\begin{aligned}
& \int_0^{0.5} (1+x^6)^{3/2}\,dx \\
= \;& \int_0^{0.5} \left(1+\frac{3}{2}x^6+\frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2!}(x^6)^2+\frac{\frac{3}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(x^6)^3+\cdots\right)dx \\
= \;& \int_0^{0.5} \left(1+\frac{3}{2}x^6+\frac{3}{8}x^{12}-\frac{1}{16}x^{18}+\cdots\right)dx \\
= \;& \left[x+\frac{3}{14}x^7+\frac{3}{104}x^{13}-\frac{1}{304}x^{19}+\cdots\right]_0^{0.5} \\
= \;& 0.5+0.001674107143+0.000003521259014-6.274\times 10^{-9}+\ldots \\
= \;& 0.5016776284\ldots
\end{aligned}$$

which is correct to within $6.274\times 10^{-9} < 10^{-6}$, by the alternating series remainder term. Note: the first three terms of the series are positive; after that the terms alternate minus, plus.

11. [10 marks] Find and classify the critical points of the function $f(x, y) = 2x^4 + y^2 - 4xy$.

Solution: Let $z = f(x, y)$. Then

$$\frac{\partial z}{\partial x} = 8x^3 - 4y \text{ and } \frac{\partial z}{\partial y} = 2y - 4x.$$

Critical points:

$$\begin{array}{rcl} 8x^3 - 4y & = & 0 \\ 2y - 4x & = & 0 \end{array} \Rightarrow \begin{array}{rcl} 2x^3 & = & y \\ y & = & 2x \end{array} \Rightarrow x^3 = x \text{ and } y = 2x;$$

so the critical points are

$$(x, y) = (0, 0),\ (1, 2) \text{ or } (-1, -2).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 24x^2,\ \frac{\partial^2 z}{\partial y^2} = 2,\ \frac{\partial^2 z}{\partial x \partial y} = -4,\ \text{and } \Delta = 48x^2 - 16$$

At $(x, y) = (0, 0), \Delta = -16 < 0$, so f has a saddle point at $(x, y) = (0, 0)$.

At $(x, y) = (1, 2)$ or $(-1, -2), \Delta = 32 > 0$ and $\dfrac{\partial^2 z}{\partial y^2} = 2 > 0$, so f has a minimum value at both $(x, y) = (1, 2)$ or $(-1, -2)$.

12. [10 marks] The acceleration of a Maserati sports car is proportional to the difference between 250 km/hr and its velocity v. If this sports car can accelerate from rest to 100 km/hr in 10 sec, how long will it take for the car to accelerate to 200 km/hr?

Solution: Let v be the velocity of the car at time t. We are given that

$$a = \frac{dv}{dt} = K(250 - v),$$

for some proportionality constant K.

Solve by separating variables:

$$\begin{aligned} \frac{dv}{dt} = K(250-v) \quad &\Rightarrow \quad \int \frac{dv}{250-v} = \int K\,dt \\ (\text{ for } v < 250 \text{ }) \quad &\Rightarrow \quad -\ln(250-v) = Kt + C \end{aligned}$$

To find C, let $t = 0$ and $v = 0$, so $C = -\ln 250$. To find K, let $t = 10$ and $v = 100$, so

$$-\ln(250 - 100) = 10K - \ln 250 \Leftrightarrow K = \frac{1}{10} \ln\left(\frac{5}{3}\right).$$

Finally, let $v = 200$ and solve for t :

$$-\ln(250 - 200) = \frac{t}{10} \ln\left(\frac{5}{3}\right) - \ln 250 \Leftrightarrow t = \frac{10 \ln 5}{\ln(5/3)} \simeq 31.5$$

So it takes about 31.5 seconds for the car to accelerate from zero to 200 km/hr.

13. [10 marks] Find the fifth degree Taylor polynomial of $f(x) = e^{\sin x}$ at $a = 0$.

Solution: using brute force. Not recommended.

$$f'(x) = \cos(x)\, e^{\sin x};\, f''(x) = -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x};$$
$$f^{(3)}(x) = -\cos(x)\, e^{\sin x} - 3\sin x\, \cos x\, e^{\sin x} + \cos^3 x\, e^{\sin x};$$
$$f^{(4)}(x) = \sin x\, e^{\sin x} - 4\cos^2 x\, e^{\sin x} + 3\sin^2 x\, e^{\sin x} - 6\sin x \cos^2 x\, e^{\sin x} + \cos^4 x\, e^{\sin x};$$

$$\begin{aligned} f^{(5)}(x) &= \cos x\, e^{\sin x} + 15\sin x \cos x\, e^{\sin x} - 10\cos^3 x\, e^{\sin x} + 15\sin^2 x\, \cos x\, e^{\sin x} - \\ &\quad 10\sin x \cos^3 x\, e^{\sin x} + \cos^5 x\, e^{\sin x}. \end{aligned}$$

So
$$f(0) = 1;\, f'(0) = 1;\, f^{(2)}(0) = 1;\, f^{(3)}(0) = 0;\, f^{(4)}(0) = -3;\, f^{(5)}(0) = -8$$

and
$$P_5(x) = 1 + x + \frac{1}{2}x^2 - \frac{3}{24}x^4 - \frac{8}{120}x^5 = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{15}x^5.$$

Alternate Solution: using power series freely, and keeping track only of powers of x up to 5.

$$\begin{aligned} e^{\sin x} &= e^{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)} \\ &= e^x e^{-x^3/6} e^{x^5/120} \cdots \\ &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right)\left(1 - \frac{x^3}{6} + \cdots\right)\left(1 + \frac{x^5}{120} + \cdots\right)\cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots - \frac{x^3}{6} - \frac{x^4}{6} - \frac{x^5}{12} - \cdots + \frac{x^5}{120} + \cdots \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{15}x^5 + \cdots \end{aligned}$$

So as before,
$$P_5(x) = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{15}x^5.$$