University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2008**
First Year - CHE, CIV, IND, LME, MEC, MMS

MAT 187H1S - CALCULUS II
Exam Type: A

Comments: the results on this exam were disappointing.

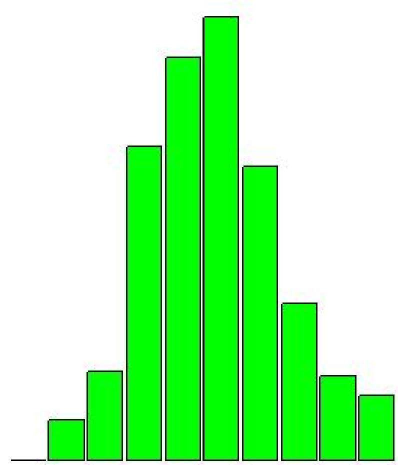
1. The average on the multiple choice part was 16/24.
2. Questions 7(a), 7(b), 8, 9, 11 and 12 are all completely routine questions. They should have been aced!
3. 7(c) was based on homework question #35 of Section 10.6.
4. Only questions 10 and 13 required any real thought.

Alternate Solutions:

1. 7(a) converges by the ratio test or the alternating series test, as well.
2. 7(b) converges by the comparison test or the limit comparison test, as well.

Breakdown of Results: 469 students wrote this exam. The marks ranged from 13% to 98%, and the average was 52.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.4%
A	7.9%	80-89%	4.5%
B	8.3%	70-79%	8.3%
C	15.6%	60-69%	15.6%
D	23.4%	50-59%	23.4%
F	44.8%	40-49%	21.3%
		30-39%	16.6%
		20-29%	4.7%
		10-19%	2.1%
		0-9%	0.0%



1. What is the third degree Taylor polynomial of the function $f(x) = \dfrac{1}{1+x}$ at $a = 0$?

(a) $1 + x + x^2 + x^3$

(b) $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

(c) $1 - x + x^2 - x^3$

(d) $1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$

Solution:

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= 1 - x + x^2 - x^3 + \cdots \\ \Rightarrow P_3(x) &= 1 - x + x^2 - x^3 \end{aligned}$$

The answer is (c).

2. The length of the polar curve with polar equation $r = e^{-\theta}$ for $\theta \geq 0$ is

(a) 0

(b) 1

(c) $\sqrt{2}$

(d) ∞

Solution:

$$\begin{aligned} \int_0^\infty \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^\infty \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta \\ &= \sqrt{2} \int_0^\infty e^{-\theta}\, d\theta \\ &= \sqrt{2} \lim_{b\to\infty} \left[-e^{-\theta}\right]_0^b = \sqrt{2} \end{aligned}$$

The answer is (c).

3. What is the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{3^n}(x-1)^n$?

(a) 1

(b) $\dfrac{1}{3}$

(c) $\ln 3$

(d) 3

Solution:

$$\begin{aligned} a_n = \frac{\ln n}{3^n} \Rightarrow R &= \lim_{n\to\infty} \frac{a_n}{a_{n+1}} \\ &= \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \frac{3^{n+1}}{3^n} \\ &= 3 \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = 3 \cdot 1 = 3 \end{aligned}$$

The answer is (d).

4. If the position vector of a particle at time t is given by $\mathbf{r} = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + 4\tan^{-1} t\,\mathbf{k}$, then its speed at time $t = 1$ is

(a) 3

(b) $\sqrt{1+\pi^2}$

(c) $\sqrt{3}$

(d) $\dfrac{\pi}{3}$

Solution: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 2t\,\mathbf{i} + \dfrac{1}{t}\,\mathbf{j} + \dfrac{4}{1+t^2}\,\mathbf{k}$.
At $t = 1$,

$$\mathbf{v} = 2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$$

and the speed is $|\mathbf{v}| = \sqrt{2^2 + 1^2 + 2^2} = 3$.
The answer is (a).

5. The area of the region inside the cardioid with equation $r = 2 + 2\cos\theta$ but outside the circle with equation $r = 2$ is given by

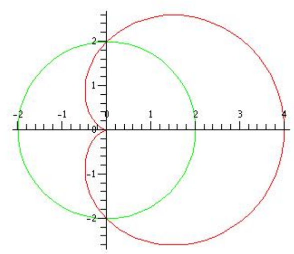
(a) $\displaystyle\int_0^{\pi/2} [2\cos\theta]^2\,d\theta$

(b) $\displaystyle\int_0^{\pi/2} [(2+2\cos\theta)^2 - 2^2]\,d\theta$

(c) $\displaystyle\int_{\pi/2}^{\pi} [2\cos\theta]^2\,d\theta$

(d) $\displaystyle\int_{\pi/2}^{\pi} [(2+2\cos\theta)^2 - 2^2]\,d\theta$

Solution:



$$A = 2\left(\frac{1}{2}\int_0^{\pi/2} [(2+2\cos\theta)^2 - 2^2]\,d\theta\right)$$

The answer is (b).

6. How many inflection points are there on the curve with parametric equations

$$x = t^2 + 4t;\, y = t^3 - 3t?$$

Recall: as in Calculus I, an inflection point on a curve is a point where the concavity changes.

(a) 0

(b) 1

(c) 2

(d) 3

Solution:

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t+4};\ \frac{d^2y}{dx^2} = \frac{6t^2 + 24t + 6}{(2t+4)^3}.$$

There are inflection points at $t = -2, t = -2 \pm \sqrt{3}$.
The answer is (d).

7. [12 marks; 4 for each part.] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\sum_{n=1}^{\infty} \left(\frac{-3n}{4n+1} \right)^n$ ⊗ Converges ○ Diverges

by the root test.

Calculation:

$$\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{-3n}{4n+1} \right)^n \right|} = \lim_{n\to\infty} \frac{3n}{4n+1} = \frac{3}{4} < 1.$$

(b) $\sum_{n=0}^{\infty} \frac{\tan^{-1} n}{n^2+1}$ ⊗ Converges ○ Diverges

by the integral test

Calculation:

$$u = \tan^{-1} x \Rightarrow \int_0^{\infty} \frac{\tan^{-1} x}{1+x^2}\, dx = \int_0^{\pi/2} u\, du < \infty.$$

(c) $\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{n+1}}$ ○ Converges ⊗ Diverges

by the limit comparison test

Calculation:

$$a_n = \frac{n^n}{(n+1)^{n+1}}; b_n = \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n = e^{-1}$$

and the series $\sum b_n$ diverges (by the integral test.) Note: both the ratio test and the root test will fail.

8. [12 marks] The displacement, $x(t)$, of an underdamped mass-spring system satisfies

$$x''(t) + 2x'(t) + 65x(t) = 0; x(0) = -2 \text{ and } x'(0) = 4.$$

Solve for x as a function of t and sketch its graph for $0 \leq t \leq \pi$, indicating both its pseudo period and its time-varying amplitude.

Solution: the auxiliary quadratic is $r^2 + 2r + 65$. Solve:

$$r^2 + 2r + 65 = 0 \Leftrightarrow r = \frac{-2 \pm \sqrt{4 - 260}}{2} = -1 \pm 8i.$$

Thus

$$x = C_1 e^{-t} \cos(8t) + C_2 e^{-t} \sin(8t).$$

To find C_1 use the initial condition $x = -2$ when $t = 0$:

$$-2 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = -2.$$

To find C_2 you need to find $x'(t)$:

$$x' = C_1(-e^{-t} \cos(8t) - 8e^{-t} \sin(8t)) + C_2(-e^{-t} \sin(8t) + 8e^{-t} \cos(8t)).$$

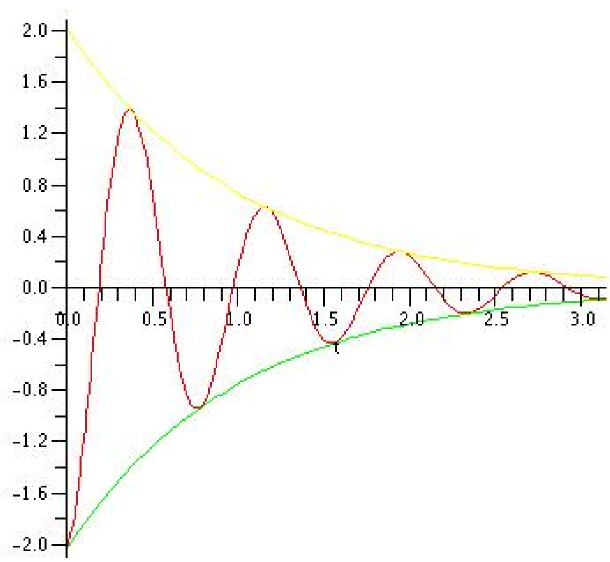
Now substitute $t = 0, x' = 4, C_1 = -2$:

$$4 = -2(-1 - 0) + C_2(0 + 8) \Leftrightarrow 8C_2 = 2 \Leftrightarrow C_2 = \frac{1}{4}.$$

Thus

$$x = -2e^{-t} \cos(8t) + \frac{1}{4} e^{-t} \sin(8t).$$

Graph:



The **pseudo period** is

$$\frac{2\pi}{8} = \frac{\pi}{4};$$

time-varying amplitude is

$$\sqrt{(-2)^2 + \left(\frac{1}{4}\right)^2} e^{-t} = \frac{\sqrt{65}}{4} e^{-t}.$$

9.[12 marks: 6 for each part.]

(a) Write down the first four non-zero terms of the Maclaurin series for each of $f(x) = e^{-x^2}$ and $g(x) = \int_0^x f(t)\,dt$.

Solution:

$$\begin{aligned} f(x) &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots \end{aligned}$$

$$\begin{aligned} g(x) &= \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots\right) dt \\ &= \left[t - \frac{t^3}{3} + \frac{t^5}{10} - \frac{t^7}{42} + \cdots\right]_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \end{aligned}$$

(b) Approximate $\int_0^{0.5} \sqrt{1+x^4}\,dx$ to within 10^{-6}, and explain why your approximation is correct to within 10^{-6}.

Solution: use the binomial theorem.

$$\begin{aligned} \int_0^{0.5} \sqrt{1+x^4}\,dx &= \int_0^{0.5} (1+x^4)^{1/2}\,dx \\ &= \int_0^{0.5} \left(1 + \frac{x^4}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 - \cdots\right) dx \\ &= \int_0^{0.5} \left(1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \cdots\right) dx \\ &= \left[x + \frac{x^5}{10} - \frac{x^9}{72} + \frac{x^{13}}{208} - \cdots\right]_0^{0.5} \\ &= 0.5 + 0.003125 - 0.00002712673611\ldots \\ &= 0.5030978733\ldots \end{aligned}$$

which is correct to within $\frac{(0.5)^{13}}{208} = 0.0000005868765\cdots < 10^{-6}$ by the alternating series remainder term.

10. [10 marks] Find $\displaystyle\int_0^\infty \frac{1}{e^{ax}(1+e^{2ax})}\,dx$, if $a>0$.

Solution: let $e^{ax}=\tan\theta$. Then

$$ae^{ax}\,dx = \sec^2\theta\,d\theta \Rightarrow dx = \frac{1}{a}\frac{1}{\tan\theta}\sec^2\theta\,d\theta.$$

Note:

$$x=0\Rightarrow \tan\theta = 1 \Rightarrow \theta=\frac{\pi}{4}$$

and since $a>0$,

$$x\to\infty \Rightarrow \tan\theta\to\infty \Rightarrow \theta\to\frac{\pi}{2}^-.$$

Thus

$$\begin{aligned}
\int_0^\infty \frac{1}{e^{ax}(1+e^{2ax})}\,dx &= \int_{\pi/4}^{\pi/2} \frac{1}{\tan\theta(1+\tan^2\theta)}\frac{1}{a}\frac{1}{\tan\theta}\sec^2\theta\,d\theta \\
&= \frac{1}{a}\int_{\pi/4}^{\pi/2}\cot^2\theta\,d\theta \\
&= \frac{1}{a}\int_{\pi/4}^{\pi/2}(\csc^2\theta-1)\,d\theta \\
&= \frac{1}{a}\left[-\cot\theta-\theta\right]_{\pi/4}^{\pi/2} \\
&= \frac{1}{a}\left(0-\frac{\pi}{2}+1+\frac{\pi}{4}\right) \\
&= \frac{1}{a}\left(1-\frac{\pi}{4}\right)
\end{aligned}$$

Alternate Solution: let $u=e^{ax}$; then $du = ae^{ax}\,dx \Rightarrow dx = \dfrac{1}{a\,u}\,du$. Thus

$$\begin{aligned}
\int_0^\infty \frac{1}{e^{ax}(1+e^{2ax})}\,dx &= \int_1^\infty \frac{1}{a}\frac{1}{u^2(1+u^2)}\,du \\
\text{(partial fractions)} \quad &= \frac{1}{a}\int_1^\infty\left(\frac{1}{u^2}-\frac{1}{1+u^2}\right)du \\
&= \frac{1}{a}\lim_{b\to\infty}\left[-\frac{1}{u}-\tan^{-1}u\right]_1^b \\
&= \frac{1}{a}\left(1-\frac{\pi}{4}\right),\ \text{as before}
\end{aligned}$$

11. [10 marks] Given that $-\frac{\pi}{2} < x < \frac{\pi}{2}$, find the general solution to the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = \cos x + \sin x.$$

Is there a solution that passes through the point $(x, y) = (0, 0)$?

Solution: must rearrange the equation:

$$\cos x \frac{dy}{dx} + y \sin x = \cos x + \sin x \Leftrightarrow \frac{dy}{dx} + y \tan x = 1 + \tan x.$$

Note: $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \sec x > 0$. The integrating factor is

$$\rho = e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$$

and the general solution is

$$\begin{aligned} y &= \frac{\int \rho(1 + \tan x)\, dx}{\rho} \\ &= \frac{\int (\sec x + \sec x \tan x)\, dx}{\sec x} \\ &= \frac{\ln|\sec x + \tan x| + \sec x + C}{\sec x} \end{aligned}$$

If $(x, y) = (0, 0)$, then

$$0 = \frac{\ln(\sec 0 + \tan 0) + \sec 0 + C}{\sec 0} \Leftrightarrow C = -1;$$

so, yes, there is a solution to the differential equation that passes through the point $(x, y) = (0, 0)$.

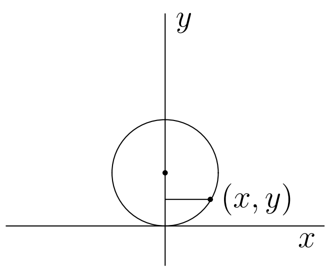
12. [10 marks] Torricelli's Law states that

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

where y is the depth of a fluid in a tank at time t, $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and g is the acceleration due to gravity.

A spherical water tank of radius 1 m is initially full. At 12 noon a plug at the bottom of the tank is removed, and 20 min later the tank is half empty. When will the tank be completely empty?

Solution:



$$x^2 + (y-1)^2 = 1$$

$$\begin{aligned} A(y) &= \pi x^2 \\ &= \pi\left(1-(y-1)^2\right) \\ &= \pi(1-y^2+2y-1) \\ &= \pi(2y-y^2) \end{aligned}$$

Solve the DE by separating variables:

$$\begin{aligned} A(y)\frac{dy}{dt} = -a\sqrt{2gy} \quad &\Leftrightarrow \quad \pi\int \frac{2y-y^2}{\sqrt{y}}\,dy = -\pi\int K\,dt, \text{ for } K = \frac{a\sqrt{2g}}{\pi} \\ &\Leftrightarrow \quad \int \left(2\sqrt{y} - y^{3/2}\right)\,dy = -\int K\,dt \\ &\Leftrightarrow \quad \frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -Kt + C, \text{ for some } C \end{aligned}$$

Let t be measured in minutes; let $t = 0$ be noon. When $t = 0, y = 2$, so

$$\frac{4}{3}2^{3/2} - \frac{2}{5}2^{5/2} = C \Leftrightarrow C = \frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2} = \frac{16}{15}\sqrt{2}.$$

When $t = 20, y = 1$, so

$$\frac{4}{3} - \frac{2}{5} = -20K + \frac{16}{15}\sqrt{2} \Leftrightarrow 20K = \frac{16}{15}\sqrt{2} - \frac{14}{15} \Leftrightarrow K = \frac{8\sqrt{2}-7}{150}$$

The tank is empty when $y = 0$:

$$0 = -Kt + C \Leftrightarrow t = \frac{C}{K} = \frac{160\sqrt{2}}{8\sqrt{2}-7} \simeq 52.45.$$

So the tank will be empty at about 12:53 PM.

13. [10 marks] Use power series to find the Taylor series of

$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

at $a = 0$. What is its interval of convergence?

Solution: use the fact that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots, \text{ if } |x| < 1.$$

What is the domain of f? x is in the domain of f if and only if

$$\begin{aligned} \frac{1+x}{1-x} > 0 \quad &\Leftrightarrow \quad (1+x)(1-x) > 0 \\ &\Leftrightarrow \quad 1 - x^2 > 0 \\ &\Leftrightarrow \quad 1 > x^2 \\ &\Leftrightarrow \quad 1 > |x| \end{aligned}$$

Both $\ln(1+x)$ and $\ln(1-x)$ are defined for $|x| < 1$; hence

$$\begin{aligned} f(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\ &= \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) \\ &= \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)\right) \\ &= \frac{1}{2}\left(2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots\right) \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots, \text{ if } |x| < 1 \end{aligned}$$

This series diverges at $x = \pm 1$, since

$$\pm 1\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} + \cdots\right)$$

diverges by the integral test; or by using the limit comparison test for

$$a_n = \frac{1}{2n+1}, b_n = \frac{1}{n}.$$

So the interval of convergence is the open interval $(-1, 1)$.