MAT186H1F - Calculus I - Fall 2019

Solutions to Term Test 1 - October 15, 2019

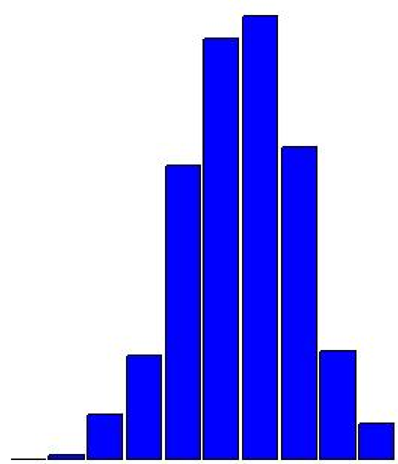
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

<u>Comments:</u>

- In Question 6(a), there is NO choice for the angle $\tan^{-1}(-2/3)$; it musts be in $[-\pi/2, 0]$. Many students used a calculator approximation for the answer even though it said, "find the exact value."
- In Question 6(b) many students ignored the instruction to use the definition of the derivative.
- In Question 8(b) the only way to calculate the limits is by rationalizing. Many students tried dividing the whole expression by x or by squaring parts of the expression—but this changes the question!
- In Question 7(a) it *is* necessary to put some restriction on x since $2x^2 - x^4/2$ is not always positive.
- Except for Question 1, which had a minimum score of 3, the range on every question was 0 to 10.

Breakdown of Results: 853 registered students wrote this test. The marks ranged from 13.75% to 97.5%, and the average was 47.5/80 or 59.4%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.0%
A	8.1%	80-89%	6.1%
B	17.7%	70-79%	17.7%
C	25.1%	60-69%	25.1%
D	23.8%	50-59%	23.8%
F	25.3%	40-49%	16.6%
		30-39%	5.9%
		20-29%	2.6%
		10-19%	0.2%
		0-9%	0.0%



1. [avg: 8.96/10] Find the following:

   (a) [3 marks] $\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9}$

   Solution: the limit is in the 0/0 form. Try factoring.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9} = \lim_{x \to -3} \frac{(x+3)(x-2)}{(x-3)(x+3)} = \lim_{x \to -3} \frac{(x-2)}{(x-3)} = \frac{-5}{-6} = \frac{5}{6}$$

   (b) [3 marks] $\frac{d(x\, e^{-x})}{dx}$

   Solution: use the product rule.

$$\frac{d(x\, e^{-x})}{dx} = 1 \cdot e^{-x} + x\,(-e^{-x}) = e^{-x} - x\, e^{-x}.$$

   (c) [4 marks] the slope of tangent line to the graph of $y = \sin(x^2)$ at $x = \sqrt{\frac{\pi}{4}}$.

   Solution: by the chain rule,

$$\frac{dy}{dx} = 2x\, \cos(x^2).$$

   At $x = \sqrt{\frac{\pi}{4}}$,

$$\frac{dy}{dx} = 2\sqrt{\frac{\pi}{4}} \cos\left(\frac{\pi}{4}\right) = \sqrt{\pi}\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}}.$$

   So the slope of tangent line to the graph of $y = \sin(x^2)$ at $x = \sqrt{\frac{\pi}{4}}$ is $\sqrt{\frac{\pi}{2}}$.

2. [avg: 4.07/10] Let $f(x) = 2^x$. Also, assume that for $r \neq 1$, the formula for the sum of a finite geometric series is

$$1 + r + r^2 + \cdots + r^{n-1} = \sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r - 1}.$$

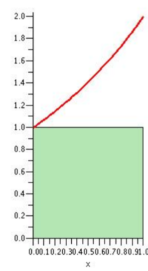
- Let L_n denote the Riemann sum approximation of the area under the curve $y = f(x)$ on the interval $[0, 1]$ using n subintervals of equal length and the *left* endpoint of each subinterval.
- Let R_n denote the Riemann sum approximation of the area under the curve $y = f(x)$ on the interval $[0, 1]$ using n subintervals of equal length and the *right* endpoint of each subinterval.

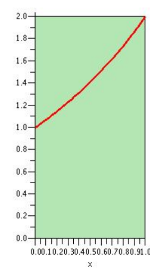
(a) [5 marks] Without calculating, arrange the numbers $L_1, R_1, L_{2019}, R_{2019}, \int_0^1 2^x\, dx$ from smallest to largest.

Solution: since $f(x) = 2^x$ is an increasing function,

$$L_1 < L_{2019} < \int_0^1 2^x\, dx < R_{2019} < R_1.$$

Aside: see figures to the right for the case $n = 1$. But no justification is required for this question.





(b) [5 marks] Calculate L_{100} and R_{100}. (Make use of the formula for the sum of a finite geometric series, as given above.) Then use your calculator to find the approximate values of L_{100} and R_{100} to 4 decimal places.

Solution: let $\Delta x = \frac{1-0}{100} = \frac{1}{100}$; let $x_i = \frac{i}{100}$; let $r = 2^{1/100}$.

$$L_{100} = \sum_{i=0}^{99} f(x_i)\Delta x = \frac{1}{100}\sum_{i=0}^{99} 2^{i/100} = \frac{1}{100}\sum_{i=0}^{99} r^i = \frac{1}{100}\left(\frac{r^{100}-1}{r-1}\right) = \frac{1}{100}\left(\frac{2-1}{2^{1/100}-1}\right);$$

that is,

$$L_{100} = \frac{0.01}{2^{1/100}-1} \approx 1.4377$$

Similarly, but a little trickier,

$$R_{100} = \sum_{i=1}^{100} f(x_i)\Delta x = \frac{1}{100}\sum_{i=1}^{100} 2^{i/100} = \frac{1}{100}\sum_{i=1}^{100} r^i = \frac{1}{100}\left(\frac{r^{101}-1}{r-1} - 1\right) = \frac{1}{100}\left(\frac{2^{1.01}-2^{1/100}}{2^{1/100}-1}\right);$$

that is,

$$R_{100} = \frac{1}{100}\left(\frac{2^{1.01}-2^{1/100}}{2^{1/100}-1}\right) \approx 1.4477$$

3. [avg: 3.92/10] Consider the function

$$f(x) = \frac{|x+2|(x-7)}{(x-c)}.$$

(a) [1 mark] For which values of x is $f(x)$ continuous?

Solution: $x \neq c$

(b) [3 marks] For which value(s) of c does $f(x)$ have a removable discontinuity?

Solution: if $c = 7$ then $f(x)$ has a removable discontinuity at $x = 7$ since

$$\lim_{x\to 7} f(x) = \lim_{x\to 7} \frac{|x+2|(x-7)}{(x-7)} = \lim_{x\to 7} |x+2| = 9;$$

that is, $\lim_{x\to 7} f(x)$ exists, even though $f(7)$ does not exist.

(c) [3 marks] For which value(s) of c does $f(x)$ have a jump discontinuity?

Solution: if $c = -2$ then $f(x)$ has a jump discontinuity at $x = -2$ since

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{|x+2|(x-7)}{(x+2)} = \lim_{x\to -2^+} \frac{(x+2)(x-7)}{(x+2)} = \lim_{x\to -2^+} (x-7) = -9$$

but

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{|x+2|(x-7)}{(x+2)} = \lim_{x\to -2^-} \frac{-(x+2)(x-7)}{(x+2)} = \lim_{x\to -2^+} -(x-7) = +9;$$

that is, both $\lim_{x\to -2^+} f(x)$ and $\lim_{x\to -2^-} f(x)$ exist but are not the same.

(d) [3 marks] For which value(s) of c does $f(x)$ have an infinite discontinuity?

Solution: if $c \neq -2, 7$ then $f(x)$ has an infinite discontinuity at $x = c$ since

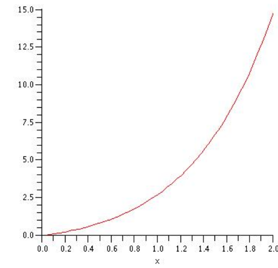
$$\lim_{x\to c^+} f(x) = \lim_{x\to c^+} \frac{|x+2|(x-7)}{(x-c)} = \begin{cases} +\infty, & \text{if } c > 7 \\ -\infty, & \text{if } c < 7, c \neq -2 \end{cases}$$

OR

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^-} \frac{|x+2|(x-7)}{(x-c)} = \begin{cases} -\infty, & \text{if } c > 7 \\ +\infty, & \text{if } c < 7, c \neq -2 \end{cases}$$

Note: you only have to show one of the one-sided limits is infinite, not both. But in either case, whether the limit is $+\infty$ or $-\infty$ depends on c, and that must be spelled out.

4. [avg: 5.58/10] Consider the function $f(x) = xe^x$ restricted to the domain $x \geq 0$. (See the graph to the right.) The Lambert W function is defined as the inverse of the function f,



$$W(x) = f^{-1}(x).$$

(a) [2 marks] What is the domain of W?

Solution: the domain of W is the range of f, which is $[0, \infty)$.

(b) [3 marks] Let $a \geq 1$. Show that $W(a \ln(a)) = \ln(a)$.

Solution: since $W = f^{-1}$, we have $W(a \ln(a)) = \ln(a) \Leftrightarrow a \ln a = f(\ln a)$. This last equation can be checked directly:

$$f(\ln a) = (\ln a)(e^{\ln a}) = (\ln a)(a) = a \ln a.$$

Aside: the domain of $f(x)$ is $x \geq 0$; that is why $a \geq 1$ is necessary—to make $\ln a \geq 0$.

(c) [3 marks] Explain why the equation $x^x = 2$ has a solution in the interval $[1, 2]$. Name any theorem you are using.

Solution: let $g(x) = x^x$, which we assume is continuous for $x > 0$. Then

$$g(1) = 1 < 2 < 4 = g(2).$$

So by the Intermediate Value Theorem there is a number $c \in (1, 2)$ such that $g(c) = 2 \Leftrightarrow c^c = 2$.

(d) [2 marks] Find an explicit solution to the equation $x^x = 2$ in terms of the Lambert W function.

Solution:

$$\begin{aligned}
x^x = 2 \quad &\Rightarrow \quad \ln(x^x) = \ln 2 \\
&\Rightarrow \quad x \ln x = \ln 2 \\
&\Rightarrow \quad W(x \ln x) = W(\ln 2) \\
\text{(use part b)} \quad &\Rightarrow \quad \ln x = W(\ln 2) \\
&\Rightarrow \quad x = e^{W(\ln 2)}
\end{aligned}$$

5. [avg: 7.78/10] For an athlete running a 50-meter dash, the position of the athlete is given by $s(t) = \dfrac{t^3}{6} + 4t$, where $s(t)$ is the athlete's distance in meters from the starting line after t seconds.

(a) [4 marks] Explain why it will take the athlete between 5 and 6 seconds to run the race. Name any theorem you are using.

Solution: s is a continuous function and

$$s(5) = \frac{245}{6} < 50 < 60 = s(6).$$

By the Intermediate Value Theorem $s(t) = 50$ for some $t \in (5, 6)$. Thus the athlete will reach the finish line sometime between 5 and 6 seconds.

Aside: this *assumes* that s is an increasing function, which it is since $s'(t) > 0$, and that there is no $t < 5$ such that $s(t) = 50$.

(b) [3 marks] What is the athlete's average speed for the time interval $1.9 \leq t \leq 2.1$? Approximate your answer to four decimal places.

Solution: calculate

$$\frac{s(2.1) - s(1.9)}{2.1 - 1.9} = \frac{7.202}{1.2} \approx 6.0017$$

So the athlete's average speed for $1.9 \leq t \leq 2.1$ is about 6.0017 m/sec.

(c) [3 marks] What is the athlete's instantaneous speed at $t = 2$?

Solution: we have instantaneous speed given by

$$s'(t) = \frac{t^2}{2} + 4.$$

So the athlete's instantaneous speed at $t = 2$ is

$$s'(2) = 2 + 4 = 6$$

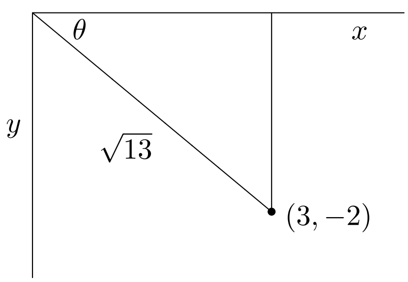
m/sec.

6. [avg: 6.77/10]

6. (a) [5 marks] Find the exact value of $\cos\left(2\tan^{-1}\left(-\frac{2}{3}\right)\right)$.

Solution: let $\theta = \tan^{-1}\left(-\frac{2}{3}\right)$. By definition of $\tan^{-1}$ we know $-\pi/2 < \theta < 0$.

We have $\tan\theta = -2/3$, $\sin\theta < 0$ and $\cos\theta > 0$. You can use either $\sin\theta$ OR $\cos\theta$:



- In the diagram to the left $\tan\theta = -2/3$.
- The length of the radius is $\sqrt{13}$.
- Then

$$\sin\theta = -\frac{2}{\sqrt{13}}$$

or

$$\cos\theta = \frac{3}{\sqrt{13}}.$$

Then the exact value of $\cos\left(2\tan^{-1}\left(-\frac{2}{3}\right)\right)$ is given by

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{3}{\sqrt{13}}\right)^2 - 1 = \frac{5}{13},$$

OR

$$\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(-\frac{2}{\sqrt{13}}\right)^2 = \frac{5}{13}.$$

OR

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{3}{\sqrt{13}}\right)^2 - \left(-\frac{2}{\sqrt{13}}\right)^2 = \frac{5}{13}.$$

6. (b) [5 marks] Let $f(x) = \frac{1}{\sqrt{x+4}}$. Use the limit definition of the derivative to find $f'(5)$.

Solution: $f(5) = \frac{1}{\sqrt{9}} = \frac{1}{3}$; so

$$\begin{aligned}
f'(5) &= \lim_{h\to 0}\frac{f(h+5)-f(5)}{h} = \lim_{h\to 0}\frac{1/\sqrt{h+9}-1/3}{h} \\
&= \lim_{h\to 0}\frac{3-\sqrt{h+9}}{3h\sqrt{h+9}} = \lim_{h\to 0}\frac{(3-\sqrt{h+9})(3+\sqrt{h+9})}{3h\sqrt{h+9}\,(3+\sqrt{h+9})} \\
&= \lim_{h\to 0}\frac{9-(h+9)}{3h\sqrt{h+9}\,(3+\sqrt{h+9})} = \lim_{h\to 0}\frac{-h}{3h\sqrt{h+9}\,(3+\sqrt{h+9})} \\
&= \lim_{h\to 0}\frac{-1}{3\sqrt{h+9}\,(3+\sqrt{h+9})} = -\frac{1}{54}
\end{aligned}$$

7. [avg: 4.33/10]

7.(a) [6 marks] It can be proved that $t - \dfrac{t^2}{2} \leq \ln(1+t) \leq t$, for $t \geq 0$. Make use of this result to find

$$\lim_{x \to 0^+} \frac{\sqrt{\ln(1+x^2)+x^2}}{x}.$$

Name any other theorem you are using.

Solution: let $t = x^2 \geq 0$. Then

$$\begin{aligned}
x^2 - \frac{x^4}{2} \leq \ln(1+x^2) \leq x^2 \quad &\Rightarrow \quad 2x^2 - \frac{x^4}{2} \leq \ln(1+x^2) + x^2 \leq 2x^2 \\
\text{(say for } 0 < x < 1) \quad &\Rightarrow \quad \sqrt{2x^2 - \frac{x^4}{2}} \leq \sqrt{\ln(1+x^2)+x^2} \leq \sqrt{2x^2} \\
&\Rightarrow \quad \frac{\sqrt{2x^2 - \frac{x^4}{2}}}{x} \leq \frac{\sqrt{\ln(1+x^2)+x^2}}{x} \leq \frac{\sqrt{2x^2}}{x} \\
&\Rightarrow \quad \sqrt{2 - \frac{x^2}{2}} \leq \frac{\sqrt{\ln(1+x^2)+x^2}}{x} \leq \sqrt{2}
\end{aligned}$$

Now both

$$\lim_{x \to 0^+} \sqrt{2 - \frac{x^2}{2}} = \sqrt{2} \text{ and } \lim_{x \to 0^+} \sqrt{2} = \sqrt{2},$$

so by the Squeeze Law we can conclude that

$$\lim_{x \to 0^+} \frac{\sqrt{\ln(1+x^2)+x^2}}{x} = \sqrt{2}.$$

7.(b) [4 marks] Find $\displaystyle\lim_{x \to 0} \frac{\sin^3(2x)}{\tan^3(4x)}$.

Solution: limit is in the 0/0 form. Note that $\tan(4x) = \dfrac{\sin(4x)}{\cos(4x)}$, and that $\cos 0 = 1$. Thus

$$\lim_{x \to 0} \frac{\sin^3(2x)}{\tan^3(4x)} = \left(\lim_{x \to 0} \frac{\sin(2x)}{\tan(4x)}\right)^3 = \left(\lim_{x \to 0} \frac{\sin(2x)}{\sin(4x)} \cdot \cos(4x)\right)^3 = \left(\lim_{x \to 0} \frac{\sin(2x)}{\sin(4x)}\right)^3 \cdot 1 = \left(\lim_{x \to 0} \frac{\sin(2x)}{\sin(4x)}\right)^3.$$

Approach 1: use double angle formula.

$$\left(\lim_{x \to 0} \frac{\sin(2x)}{\sin(4x)}\right)^3 = \left(\lim_{x \to 0} \frac{\sin(2x)}{2\sin(2x)\cos(2x)}\right)^3 = \left(\lim_{x \to 0} \frac{1}{2\cos(2x)}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Approach 2: use basic trig limit.

$$\left(\lim_{x \to 0} \frac{\sin(2x)}{\sin(4x)}\right)^3 = \left(\lim_{x \to 0} \frac{2}{4} \frac{\sin(2x)}{2x} \frac{4x}{\sin(4x)}\right)^3 = \left(\frac{1}{2}\right)^3 \left(\lim_{h \to 0} \frac{\sin h}{h}\right)^3 \left(\lim_{k \to 0} \frac{k}{\sin k}\right)^3 = \frac{1}{8}(1^3)(1^3) = \frac{1}{8}.$$

Either way, the answer is $\dfrac{1}{8}$.

8. [avg: 6.09/10] Consider the function $f(x) = \sqrt{x^2 + 6x + 5}$.

(a) [2 marks] Find the domain of f.

Solution: $x^2 + 6x + 5 \geq 0 \Leftrightarrow (x+5)(x+1) \geq 0 \Leftrightarrow x \leq -5$ or $x \geq -1$. So the domain of f is, in interval notation, $(-\infty, -5] \cup [-1, \infty)$.

(b) [6 marks] A line with equation $y = mx + b$ is a *slant asymptote* to the graph of $y = f(x)$ if

$$\lim_{x\to\infty} \big[f(x) - (mx+b)\big] = 0 \quad \text{or} \quad \lim_{x\to-\infty} \big[f(x) - (mx+b)\big] = 0.$$

Show that $f(x)$ has slant asymptotes with equations $y = x + 3$ and $y = -x - 3$.

Solution: show $\lim\limits_{x\to\infty} (f(x) - (x+3)) = 0$ and $\lim\limits_{x\to-\infty} (f(x) - (-x-3)) = 0:$

$$\begin{aligned}
\lim_{x\to\infty} \left[\sqrt{x^2+6x+5} - (x+3)\right] &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+6x+5} - (x+3)\right)\left(\sqrt{x^2+6x+5} + (x+3)\right)}{\sqrt{x^2+6x+5} + (x+3)} \\
&= \lim_{x\to\infty} \frac{x^2+6x+5-x^2-6x-9}{\sqrt{x^2+6x+5} + x + 3} \\
&= \lim_{x\to\infty} \frac{-4}{\sqrt{x^2+6x+5} + x + 3} = 0;
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} \left[\sqrt{x^2+6x+5} - (-x-3)\right] &= \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+6x+5} - (-x-3)\right)\left(\sqrt{x^2+6x+5} + (-x-3)\right)}{\sqrt{x^2+6x+5} + (-x-3)} \\
&= \lim_{x\to-\infty} \frac{x^2+6x+5-x^2-6x-9}{\sqrt{x^2+6x+5} - x - 3} \\
&= \lim_{x\to-\infty} \frac{-4}{\sqrt{x^2+6x+5} - x - 3} = 0.
\end{aligned}$$

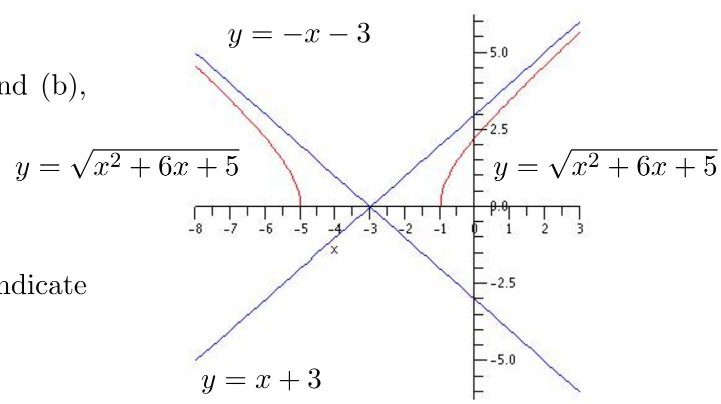
Aside: the other two limits, $\lim\limits_{x\to\infty} (f(x) - (-x-3))$ and $\lim\limits_{x\to-\infty} (f(x) - (x+3))$, are both infinite.

(c) [2 marks] Using the results from parts (a) and (b), sketch the graph of

$$y = f(x)$$

and its two slant asymptotes. Make sure to indicate the equation of each part of your sketch.

Solution: to the right.



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