**MAT186H1F - Calculus I - Fall 2018**

**Solutions to Term Test 1 - October 16, 2018**

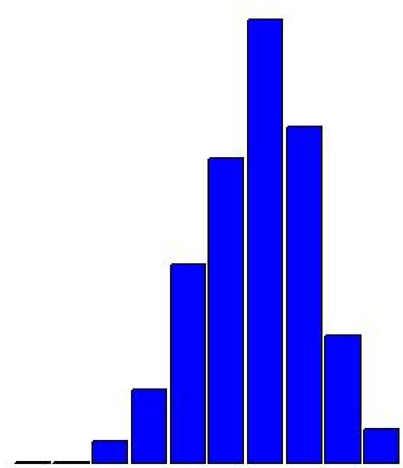
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

**Genreal Comments:**

- Statistically the results on this test are very similar to the first MAT188 Test. In particular there were two questions, #4 and #6, that both had failing averages. Interestingly both of these questions concerned inverse functions, in one form or another. While #4 was supposed to be challenging, #6 is actually routine and straightforward.
- In #8, to fully justify your choice of graph, it *is* necessary to calculate all four limits, from the left and from the right side, as $x \to 2$ and as $x \to -2$.

**Breakdown of Results:** 771 registered students wrote this test. The marks ranged from 8.75% to 98.75%, and the average was 62.2%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | % | Decade | % |
|---|---|---|---|
| | | 90-100% | 2.2% |
| A | 10.5% | 80-89% | 8.3% |
| B | 21.8% | 70-79% | 21.8% |
| C | 28.7% | 60-69% | 28.7% |
| D | 19.7% | 50-59% | 19.7% |
| F | 19.2% | 40-49% | 12.8% |
| | | 30-39% | 4.8% |
| | | 20-29% | 1.4% |
| | | 10-19% | 0.1% |
| | | 0-9% | 0.1% |



1. [avg: 7.98/10] Find the following limits, or explain why they do not exist.

(a) [4 marks] $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4}$

**Solution:** the limit is in the 0/0 form; try factoring.

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} &= \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x+2)} \\
&= \lim_{x\to 2} \frac{x+3}{x+2} \\
&= \frac{5}{4}
\end{aligned}$$

(b) [4 marks] $\lim_{x\to +\infty} \left(\sqrt{x^2+7x}-x\right)$

**Solution:** the limit is in the $\infty - \infty$ form; try rationalizing.

$$\begin{aligned}
\lim_{x\to\infty} \left(\sqrt{x^2+7x}-x\right) &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+7x}-x\right)\left(\sqrt{x^2+7x}+x\right)}{\sqrt{x^2+7x}+x} \\
&= \lim_{x\to\infty} \frac{x^2+7x-x^2}{\sqrt{x^2+7x}+x} = \lim_{x\to\infty} \frac{7x}{\sqrt{x^2+7x}+x} \\
&= \lim_{x\to\infty} \frac{7}{\frac{\sqrt{x^2+7x}}{x}+1} = \lim_{x\to\infty} \frac{7}{\sqrt{\frac{x^2+7x}{x^2}}+1} \\
&= \lim_{x\to\infty} \frac{7}{\sqrt{1+7/x}+1} \\
&= \frac{7}{\sqrt{1+0}+1} = \frac{7}{2}
\end{aligned}$$

(c) [2 marks] $\lim_{x\to 0} \frac{\sin(3x)}{5x}$

**Solution:** limit is in the 0/0 form; try reducing it to the basic trig limit, $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(3x)}{5x} &= \frac{3}{5} \lim_{x\to 0} \frac{\sin(3x)}{3x} \\
(\text{let } h = 3x) &= \frac{3}{5} \lim_{h\to 0} \frac{\sin h}{h} \\
&= \frac{3}{5} \cdot 1 = \frac{3}{5}
\end{aligned}$$

2. [avg: 8.74/10] A car comes to a stop eight seconds after the driver applies the brakes. While the brakes are on, the following speeds are recorded:

| time since brakes applied (sec) | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| speed (m/sec) | 32 | 22 | 14 | 6 | 0 |

(a) [4 marks] Use a right-endpoint Riemann sum and four equally spaced subintervals of $[0, 8]$ to approximate the distance travelled by the car.

**Solution:** $\Delta x = 2$; distance $= \int_0^8 v\,dt \approx 22 \cdot 2 + 14 \cdot 2 + 6 \cdot 2 + 0 \cdot 2 = 84.$

(b) [4 marks] Use a left-endpoint Riemann sum and four equally spaced subintervals of $[0, 8]$ to approximate the distance travelled by the car.

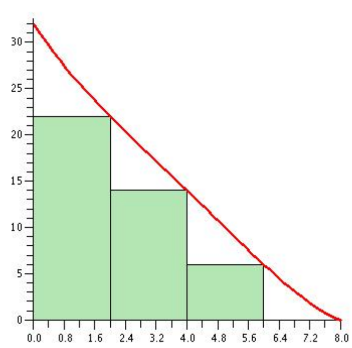
**Solution:** $\Delta x = 2$; distance $= \int_0^8 v\,dt \approx 32 \cdot 2 + 22 \cdot 2 + 14 \cdot 2 + 6 \cdot 2 = 148.$

(c) [2 marks] Assuming that the speed of the car is always decreasing on $[0, 8]$, explain why the total distance travelled by the car during the 8 seconds must be between the values of your answers from parts (a) and (b).

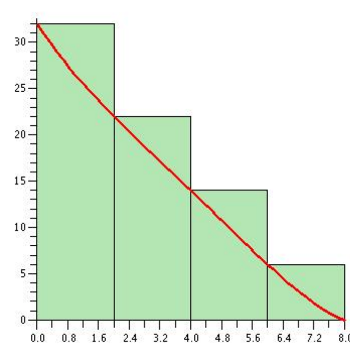
**Solution:** since the speed is decreasing, the right-endpoint Riemann sum is an under approximation to the total distance travelled, and the left-endpoint Riemann sum is an over approximation to the total distance travelled. So

$$84 < \int_0^8 v\,dt < 148.$$

Here are the plots:




(a)




(b)

3. [avg: 6.38/10] Let $g(x) = \begin{cases} 2 - 2c^2x, & \text{if } x < -1, \\ 5/2, & \text{if } x = -1, \\ 6 - 7cx^2, & \text{if } x > -1 \end{cases}$

(a) [2 marks] Find both $\lim_{x\to -1^+} g(x)$ and $\lim_{x\to -1^-} g(x)$.

**Solution:**

$$\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} (6 - 7cx^2) = 6 - 7c(-1)^2 = 6 - 7c;$$
$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^+} (2 - 2c^2x) = 2 - 2c^2(-1) = 2 + 2c^2.$$

(b) [2 marks] For which value(s) of $c$ does $\lim_{x\to -1} g(x)$ exist?

**Solution:** the two one-sided limits must be equal:

$$\lim_{x\to -1^+} g(x) = \lim_{x\to -1^-} g(x) \Leftrightarrow 6 - 7c = 2 + 2c^2 \Leftrightarrow 2c^2 + 7c - 4 = 0$$
$$\Leftrightarrow (2c-1)(c+4) = 0 \Leftrightarrow c = \frac{1}{2} \text{ or } c = -4.$$

(c) [2 marks] For which value(s) of $c$ does $g$ have a removable discontinuity at $x = -1$?

**Solution:** from part (b), if $c = -4$, then

$$\lim_{x\to -1} g(x) = 2 + 2(-4)^2 = 34 \neq f(-1) = \frac{5}{2},$$

so $g$ has a removable discontinuity at $x = -1$.

(d) [2 marks] For which value(s) of $c$ is $g$ continuous at $x = -1$?

**Solution:** from part (b), if $c = 1/2$, then

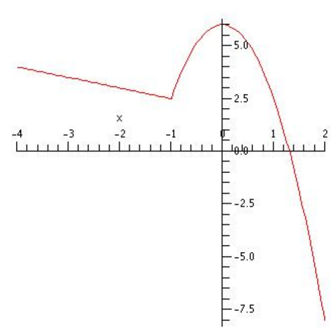
$$\lim_{x\to -1} g(x) = 6 - 7\left(\frac{1}{2}\right) = \frac{5}{2} = f(-1) = \frac{5}{2},$$

so $g$ has is continuous at $x = -1$.

(e) [2 marks] For which value(s) of $c$ is $g$ differentiable at $x = -1$?

**Solution:** to be differentiable at $x = -1$, $g$ must be continuous at $x = -1$, so we must have $c = 1/2$. But if $c = 1/2$, then

$$g(x) = \begin{cases} 2 - x/2, & \text{if } x \leq -1, \\ 6 - 7x^2/2, & \text{if } x > -1 \end{cases}.$$




The graph of $g$, to the right, shows that $g$ is not differentiable at $(x, y) = (-1, 5/2)$. So there is no value of $c$ for which $g$ is differentiable at $x = -1$.

4. [avg: 3.67/10]

Let $f$ be a function defined for all $x$ in $\mathbb{R}$ such that $f^{-1}(x) = x\exp(x^2)$; let $g(x) = \dfrac{f(x)+4}{f(x)-1}$.

(Note: it is not necessary to find $f$.)

(a) [2 mark] What is the range of $f$?

**Solution:** the range of $f$ is the domain of $f^{-1}$, which is $x \in \mathbb{R}$.

(b) [3 marks] What is the domain of $g$?

**Solution:** since $f$ is defined for all $x$, the domain of $g$ is the set of $x$ such that $f(x) \neq 1$. We have,

$$f(x) = 1 \Leftrightarrow x = f^{-1}(1) = 1 \cdot e^{(1^2)} = e.$$

So the domain of $g$ is $x \neq e$.

(c) [5 marks] Find $g^{-1}(x)$, or explain why it does not exist.

**Solution:** solve the equation $x = g(y)$ for $y$; if $y$ is a function of $x$, then you have found $g^{-1}(x)$.

$$\begin{aligned}
x = g(y) \quad &\Rightarrow \quad x = \frac{f(y)+4}{f(y)-1} \\
&\Rightarrow \quad xf(y) - x = f(y) + 4 \\
&\Rightarrow \quad xf(y) - f(y) = x + 4 \\
&\Rightarrow \quad f(y)(x-1) = x + 4 \\
&\Rightarrow \quad f(y) = \frac{x+4}{x-1} \\
&\Rightarrow \quad y = f^{-1}\left(\frac{x+4}{x-1}\right) \\
&\Rightarrow \quad y = \left(\frac{x+4}{x-1}\right)\exp\left(\left(\frac{x+4}{x-1}\right)^2\right)
\end{aligned}$$

Thus

$$g^{-1}(x) = \left(\frac{x+4}{x-1}\right)\exp\left(\left(\frac{x+4}{x-1}\right)^2\right).$$

5. [avg: 7.61/10] Suppose the amplitude at time $t$ of a seismic wave, as measured by a seismometer, is modelled by

$$y(t) = \frac{\sin(3t) + \cos(4t)}{1 + t^2}.$$

(a) [3 marks] Calculate $\lim_{t\to\infty} y(t)$.

**Solution:** use the Squeeze Law.

$$-2 \le \sin(3t) + \cos(4t) \le 2 \Rightarrow -\frac{2}{1+t^2} \le \frac{\sin(3t) + \cos(4t)}{1+t^2} \le \frac{2}{1+t^2}.$$

Since both

$$\lim_{t\to\infty} \left(-\frac{2}{1+t^2}\right) = 0 \text{ and } \lim_{t\to\infty} \frac{2}{1+t^2} = 0,$$

the Squeeze Law implies

$$\lim_{t\to\infty} \frac{\sin(3t) + \cos(4t)}{1+t^2} = 0.$$

(b) The rate of change of $y(t)$ is called the spectral velocity of the seismic wave. Find:

($i$) [2 marks] the average spectral velocity of the seismic wave on the interval $t = 0$ to $t = 0.1$; approximate your answer (with your calculator in radian mode) to four decimal places.

**Solution:**

$$\frac{\Delta y}{\Delta t} = \frac{y(0.1) - y(0)}{0.1 - 0} = 10\left(\frac{\sin 0.3 + \cos 0.4}{1.01} - 1\right) \approx 2.04535\ldots,$$

we'll accept either 2.0453 or 2.0454.

($ii$) [5 marks] the instantaneous spectral velocity of the seismic wave at $t = 0$.

**Solution:** use the quotient rule and chain rule to find

$$\frac{dy}{dt} = \frac{(3\cos(3t) - 4\sin(4t))(1+t^2) - 2t(\sin(3t) + \cos(4t))}{(1+t^2)^2};$$

at $t = 0$,

$$\frac{dy}{dt} = \frac{(3-0)(1+0) - 0}{(1+0)^2} = 3.$$

6. [avg: 3.27/10] Let $f(x) = \dfrac{1}{\tan(\sec^{-1} x)}$.

(a) [3 marks] What is the domain of $f$?

**Solution:** the domain of $\sec^{-1} x$ is $|x| \geq 1$, but if $x = 1$, then $\sec^{-1} 1 = 0$ and $\tan 0 = 0$, so $x = 1$ is not in the domain of $f$. Similarly, if $x = -1$, then $\sec^{-1}(-1) = \pi$ and $\tan \pi = 0$, so $x = -1$ also not in the domain of $f$. Thus the domain of $f$ is

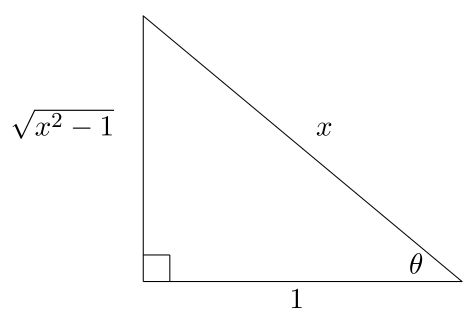
- $\{x \in \mathbb{R} \mid x < -1 \text{ or } x > 1\}$, in set notation.
- $(-\infty, -1) \cup (1, \infty)$, in interval notation.
- or simply, $|x| > 1$.

(b) [7 marks] Rewrite $f(x)$ as an expression without trigonometric functions.

**Solution:** you can use either triangles or trig identities; as illustrated in Case 1 or Case 2.
Case 1: if $x > 1$, then

$$0 < \sec^{-1} x < \frac{\pi}{2}$$

and $\tan(\sec^{-1} x) > 0$. Let $\theta = \sec^{-1} x$; then $\sec\theta = x$.




- In the triangle to the left, $x = \sec\theta$.
- The length of the other side is $\sqrt{x^2 - 1}$.
- Then

$$\tan\theta = \sqrt{x^2 - 1}.$$

Case 2: if $x < -1$, then

$$\frac{\pi}{2} < \sec^{-1} x < \pi$$

and $\tan(\sec^{-1} x) < 0$. Let $\theta = \sec^{-1} x$; then $\sec\theta = x$ and $\tan\theta < 0$. Using the trig identity, $\tan^2\theta + 1 = \sec^2\theta$, we have

$$\tan\theta = -\sqrt{\sec^2\theta - 1} = -\sqrt{x^2 - 1}.$$

Thus

$$f(x) = \frac{1}{\tan(\sec^{-1} x)} = \begin{cases} \dfrac{1}{\sqrt{x^2-1}}, & \text{if } x > 1 \\[2ex] -\dfrac{1}{\sqrt{x^2-1}}, & \text{if } x < -1 \end{cases}$$

Optionally,

$$f(x) = \frac{x}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1.$$

7. [avg: 6.62/10]

7.(a) [4 marks] State the Intermediate Value Theorem.

**Solution:** from C3,

> If $f$ is continuous on the interval $[a, b]$ and $N$ is between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there is a number $c$ in $(a, b)$ such that $f(c) = N$.

Or, from C1,

> Let $f$ be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value $y$, where $f(a) < y < f(b)$, there is at least one value $c$ in $(a, b)$ such that $f(c) = y$.

7.(b) [6 marks] Let $f(x) = 3x^{1/3} - x$. Explain, by making use of the Intermediate Value Theorem, why there are at least 3 solutions to the equation $f(x) = 1$.

**Solution:** first observe that $f$ is continuous for all $x$ in $\mathbb{R}$, so we can apply the Intermediate Value Theorem to $f$ on any closed interval $[a, b]$. We have

- $f(0) = 0 < 1$ and $f(1) = 2 > 1$, so by IVT, there is a number $c_1$ in $(0, 1)$ such that $f(c_1) = 1$.
- $f(1) = 2 > 1$ and $f(8) = -2 < 1$, so by IVT, there is a number $c_2$ in $(1, 8)$ such that $f(c_2) = 1$.
- $f(0) = 0 < 1$ and $f(-8) = 2 > 1$, so by IVT, there is a number $c_3$ in $(-8, 0)$ such that $f(c_3) = 1$.

Thus there are at least 3 distinct solutions, $c_1, c_2, c_3$, to the equation $f(x) = 1$.

8. [avg: 5.53/10] Find all the asymptotes to the graph of the function $f(x) = \dfrac{x^3 + x^2 - 6x - 4}{x^2 - 4}$, and then pick the figure below that best matches the graph of $y = f(x)$, along with its asymptotes.

**Solution:** do long division first.

$$f(x) = \frac{x^3 + x^2 - 6x - 4}{x^2 - 4} = x + 1 - \frac{2x}{x^2 - 4}.$$

Thus the line with equation $y = x + 1$ is a **slant** asymptote to the graph of $y = f(x)$.

There are no **horizontal** asymptotes, since $\displaystyle\lim_{x \to \pm\infty} \frac{x^3 + x^2 - 6x - 4}{x^2 - 4} = \pm\infty$.

For **vertical** asymptotes, check for infinite limits at $x = -2$ and $x = 2$ :

$$\lim_{x \to 2^+} \left( x + 1 - \frac{2x}{x^2 - 4} \right) = -\infty, \text{ since: } x \to 2^+ \Rightarrow x^2 - 4 \to 0^+ \text{ and } -2x \to -4;$$

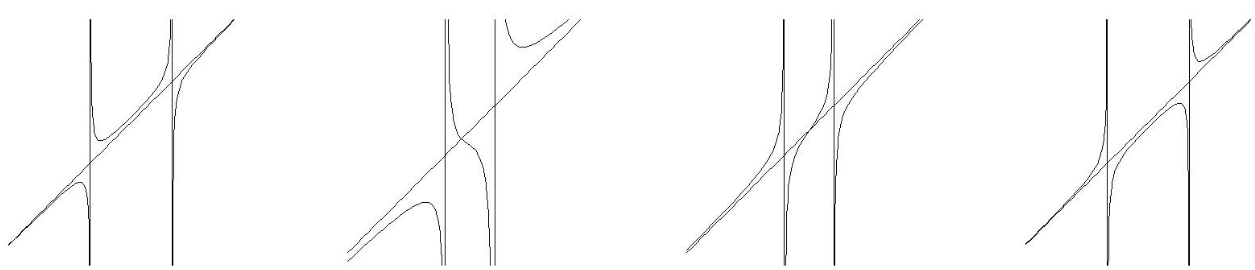
$$\lim_{x \to 2^-} \left( x + 1 - \frac{2x}{x^2 - 4} \right) = +\infty, \text{ since: } x \to 2^- \Rightarrow x^2 - 4 \to 0^- \text{ and } -2x \to -4;$$

$$\lim_{x \to -2^+} \left( x + 1 - \frac{2x}{x^2 - 4} \right) = -\infty, \text{ since: } x \to -2^+ \Rightarrow x^2 - 4 \to 0^- \text{ and } -2x \to 4;$$

$$\lim_{x \to -2^-} \left( x + 1 - \frac{2x}{x^2 - 4} \right) = +\infty, \text{ since: } x \to -2^- \Rightarrow x^2 - 4 \to 0^+ \text{ and } -2x \to 4.$$

So both $x = -2$ and $x = 2$ are equations of the vertical asymptotes to the graph of $y = f(x)$.

In terms of the figures below, pick C. (Note: you need to calculate all four of the above limits to justify your choice of C.)



A B C D

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The end.