MAT186H1F - Calculus I - Fall 2017

Solutions to Term Test 1 - October 3, 2017

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Comments:

- Question 1, parts (b)-(e) are about the definition of the inverse trig functions; nothing more.
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Breakdown of Results: 760 students wrote this test. The marks ranged from ?% to ?%, and the average was ?%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	%
A	%	80-89%	%
B	%	70-79%	%
C	%	60-69%	%
D	%	50-59%	%
F	%	40-49%	%
		30-39%	%
		20-29%	%
		10-19%	%
		0-9%	%

1. [2 marks for each part] Find all the solutions x in $\mathbb{R}$ to the following equations:

   (a) $2^{\log_2 x} = x$

   Solution: x must be in the domain of $\log_2 x$, so $x > 0$

   (b) $\sin(\sin^{-1} x) = x$

   Solution: x must be in the domain of $\sin^{-1} x$, so $-1 \leq x \leq 1$.

   (c) $\sin^{-1}(\sin x) = x$

   Solution: $y = \sin^{-1}(\sin x)$ if and only if $\sin y = \sin x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. So $y = x$ if and only if

   $$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

   (d) $x = \sec(\sec^{-1} x)$

   Solution: x must be in the domain of $\sec^{-1} x$, so $x \leq -1$ or $x \geq 1$.

   This is equivalent to saying $|x| \geq 1$.

   (e) $x = \sec^{-1}(\sec x)$

   Solution: $y = \sec^{-1}(\sec x)$ if and only if $\sec y = \sec x$ and $0 \leq y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \leq \pi$. So $y = x$ if and only if

   $$0 \leq x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x \leq \pi.$$

2. Suppose the function $f(x)$ is defined for all x in the interval $[0, 10]$ and that the following data points are on the graph of f :

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	1.1	5.7	2.2	9.0	5.2	4.3	4.6	8.8	3.2	5.7	6.9

(a) [5 marks] Approximate the value of $\displaystyle\int_0^{10} f(x)\,dx$ by using a right endpoint Riemann sum and ten equally spaced subintervals of $[0, 10]$.

Solution: take $\Delta x = 1$ and $x_i = i$. Then

$$\begin{aligned}\int_0^{10} f(x)\,dx &\approx \sum_{i=1}^{10} f(x_i)\,\Delta x \\ &= \sum_{i=1}^{10} f(i) \\ &= 5.7 + 2.2 + 9.0 + 5.2 + 4.3 + 4.6 + 8.8 + 3.2 + 5.7 + 6.9 \\ &= 55.6\end{aligned}$$

(b) [5 marks] Approximate the value of $\displaystyle\int_0^{10} f(x)\,dx$ by using a midpoint Riemann sum and five equally spaced subintervals of $[0, 10]$.

Solution: now take $\Delta x = 2$ and use only the odd x-values: that is, $i+1$ is the midpoint of the interval $[i, i+2]$, for $i = 0, 2, 4, 6, 10$. Thus

$$\begin{aligned}\int_0^{10} f(x)\,dx &\approx (f(1) + f(3) + f(5) + f(7) + f(9))\,\Delta x \\ &= (5.7 + 9.0 + 4.3 + 8.8 + 5.7)(2) \\ &= 67\end{aligned}$$

3.(a) [5 marks] Suppose $\displaystyle\int_0^2 f(x)\,dx = 10$ and $\displaystyle\int_2^3 f(x)\,dx = 3$. Find $\displaystyle\int_0^3 (f(x)+2)\,dx$.

Solution:

$$
\begin{aligned}
\int_0^3 (f(x)+2)\,dx &= \int_0^3 f(x)\,dx + \int_0^3 2\,dx \\
&= \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_0^3 2\,dx \\
&= 10 + 3 + 2\cdot 3 \\
&= 19
\end{aligned}
$$

3.(b) [5 marks] Decide if each of the following expressions represents the integral

$$\int_1^9 (x - 2\sin x)\,dx.$$

Indicate **Yes** if it does, and **No** if it doesn't.

(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8\,i}{n} - 2\sin\left(\frac{8\,i}{n}\right)\right)\left(\frac{8}{n}\right)$ ○ **Yes** ⊗ **No**

(b) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{9\,i}{n} - 2\sin\left(\frac{9\,i}{n}\right)\right)\left(\frac{9}{n}\right)$ ○ **Yes** ⊗ **No**

(c) $\displaystyle\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(1+\frac{8\,i}{n} - 2\sin\left(1+\frac{8\,i}{n}\right)\right)\left(\frac{8}{n}\right)$ ⊗ **Yes** ○ **No**

(d) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{8\,i}{n} - 2\sin\left(1+\frac{8\,i}{n}\right)\right)\left(\frac{8}{n}\right)$ ⊗ **Yes** ○ **No**

(e) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{8\,i}{n} - 2\sin\left(1+\frac{8\,i}{n}\right)\right)\left(\frac{9}{n}\right)$ ○ **Yes** ⊗ **No**

4. Consider the function $f(x) = \dfrac{10+x}{x-9}$.

(a) [3 marks] Show that f is one-to-one.

Solution: show that each value of $f(x)$ corresponds to exactly one value of x.

$$\begin{aligned} f(x_1) = f(x_2) \quad &\Rightarrow \quad \frac{10+x_1}{x_1-9} = \frac{10+x_2}{x_2-9} \\ &\Rightarrow \quad (10+x_1)(x_2-9) = (10+x_2)(x_1-9) \\ &\Rightarrow \quad x_1x_2 + 10x_2 - 9x_1 - 90 = x_2x_1 + 10x_1 - 9x_2 - 90 \\ &\Rightarrow \quad 19x_2 = 19x_1 \\ &\Rightarrow \quad x_2 = x_1 \end{aligned}$$

OR: show the graph of f passes the horizontal line test, assuming you can graph f accurately.

(b) [3 marks] Find the formula for $f^{-1}(x)$.

Solution: switch x and y in the formula for $y = f(x)$ and solve for y as a function of x.

$$x = \frac{10+y}{y-9} \Rightarrow yx - 9x = 10 + y \Rightarrow y(x-1) = 10 + 9x \Rightarrow y = \frac{10+9x}{x-1};$$

so

$$f^{-1}(x) = \frac{10+9x}{x-1}.$$

(c) [4 marks] Find the equations of the horizontal and vertical asymptotes to the graph of f^{-1}.

Solution: the vertical asymptote to the graph of f^{-1} is the line with equation $x = 1$ since

$$\lim_{x\to1^+} \frac{10+9x}{x-1} = \infty \text{ and } \lim_{x\to1^-} \frac{10+9x}{x-1} = -\infty.$$

The (only) horizontal asymptote to the graph of f^{-1} is the line with equation $y = 9$ since

$$\lim_{x\to\infty} \frac{10+9x}{x-1} = \lim_{x\to\infty} \frac{10/x+9}{1-1/x} = 9 \text{ and } \lim_{x\to-\infty} \frac{10+9x}{x-1} = \lim_{x\to-\infty} \frac{10/x+9}{1-1/x} = 9.$$

Alternate Solution: you could find the asymptotes of f and then just switch x and y. Thus:

1. the line $y = 1$ is a horizontal asymptote to the graph of f, so $x = 1$ is a vertical asymptote to the graph of f^{-1}.
2. the line $x = 9$ is a vertical asymptote to the graph of f, so $y = 9$ is a horizontal asymptote to the graph of f^{-1}.

5. Find the following limits, or explain why they do not exist.

(a) [4 marks] $\lim\limits_{x \to 4} \dfrac{x-4}{3-\sqrt{x+5}}$

Solution: the limit is of the from $0/0$. Try rationalizing the denominator.

$$\begin{aligned}
\lim_{x \to 4} \frac{x-4}{3-\sqrt{x+5}} &= \lim_{x \to 4} \left(\frac{x-4}{3-\sqrt{x+5}}\right)\left(\frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}\right) \\
&= \lim_{x \to 4} \frac{(x-4)(3+\sqrt{x+5})}{9-(x+5)} \\
&= \lim_{x \to 4} \frac{(x-4)(3+\sqrt{x+5})}{4-x} \\
&= -\lim_{x \to 4}(3+\sqrt{x+5}) \\
&= -(3+3) = -6
\end{aligned}$$

(b) [4 marks] $\lim\limits_{x \to 3} \dfrac{x^2-9}{x^2+2x-15}$

Solution: the limit is of the from $0/0$. Try factoring numerator and denominator.

$$\begin{aligned}
\lim_{x \to 3} \frac{x^2-9}{x^2+2x-15} &= \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)(x+5)} \\
&= \lim_{x \to 3} \frac{x+3}{x+5} \\
&= \frac{6}{8} = \frac{3}{4}
\end{aligned}$$

(c) [2 marks] $\lim\limits_{x \to 6} \dfrac{x-8}{(x-6)^2}$

Solution: the limit is of the form $-2/0$, so may be infinite. As $x \to 6$, the numerator is approaching -2, while denominator is approaching 0 but is always positive. So

$$\lim_{x \to 6} \frac{x-8}{(x-6)^2} = -\infty.$$

6.(a) [5 marks] Find the exact value of $\cos\left(2\sin^{-1}\left(-\frac{5}{13}\right)\right)$

Solution: use $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$. Let $\theta = \sin^{-1}\left(-\frac{5}{13}\right)$. Then $\sin\theta = -\frac{5}{13}$ and

$$\cos\left(2\sin^{-1}\left(-\frac{5}{13}\right)\right) = \cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(-\frac{5}{13}\right)^2 = 1 - \frac{50}{169} = \frac{119}{169}$$

6.(b) [5 marks] Find all the solutions x in $\mathbb{R}$ to the equation $x = -\sin(\cos^{-1}(3x+1))$

Solution: let $\theta = \cos^{-1}(3x+1)$. Then $\cos\theta = 3x+1$ and $0 \le \theta \le \pi$, by definition of the inverse cosine function. Thus $\sin\theta \ge 0$. Therefore,

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(3x+1)^2} = \sqrt{-9x^2-6x}.$$

(Aside: this may *look* incorrect, but $-9x^2 - 6x \ge 0$ since the domain of $\cos^{-1}$ is the interval $[-1, 1]$ which means

$$-1 \le 3x+1 \le 1 \Leftrightarrow -2 \le 3x \le 0 \Leftrightarrow -\frac{2}{3} \le x \le 0,$$

which in turn implies $-9x^2 - 6x \ge 0$. This is also why the negative sign is in the original equation.) Then

$$\begin{aligned}
x = -\sqrt{-9x^2-6x} \quad &\Rightarrow \quad x^2 = -9x^2 - 6x \\
&\Rightarrow \quad 10x^2 + 6x = 0 \\
&\Rightarrow \quad 5x^2 + 3x = 0 \\
&\Rightarrow \quad x(5x+3) = 0 \\
&\Rightarrow \quad x = 0 \text{ or } x = -\frac{3}{5}
\end{aligned}$$

7. Suppose the position at time t of an object moving horizontally along the x-axis is given by $x = t^2 - 4t$, for $0 \leq t \leq 8$.

(a) [3 marks] What is the average velocity of the object on the time interval $0 \leq t \leq 3$?

Solution:

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{(9-12)-0}{3-0} = -1$$

(b) [3 marks] What is the average velocity of the object on the time interval $1 \leq t \leq 1+h$?

Solution:

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{(1+h)^2 - 4(1+h) - (-3)}{h} = \frac{h^2 - 2h}{h} = h - 2$$

(c) [4 marks] What is the instantaneous velocity of the object at $t = 1$?

Solution: the instantaneous velocity of the object at $t = 1$ is defined as

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{h \to 0} \frac{x(1+h) - x(1)}{1+h-1}.$$

Thus

$$\begin{aligned} v_{instantaneous} &= \lim_{h \to 0} \frac{(1+h)^2 - 4(1+h) - (-3)}{h} \\ &= \lim_{h \to 0} (h-2), \text{ by part (b)} \\ &= -2 \end{aligned}$$

OR: you could use $x'(1) = 2(1) - 4 = -2$.

8. Find the following limits, or explain why they do not exist.

(a) [3 marks] $\lim_{x\to\infty} \frac{\cos x}{x^2}$

Solution: use the Squeeze Law, and the observation that

$$-1 \le \cos x \le 1 \Rightarrow -\frac{1}{x^2} \le \frac{\cos x}{x^2} \le \frac{1}{x^2}.$$

Since both

$$\lim_{x\to\infty} \left(-\frac{1}{x^2}\right) = 0 \text{ and } \lim_{x\to\infty} \left(\frac{1}{x^2}\right) = 0,$$

the Squeeze Law implies that also

$$\lim_{x\to\infty} \left(\frac{\cos x}{x^2}\right) = 0.$$

(b) [3 marks] $\lim_{x\to -1^-} \frac{|1-x^2|}{x(x+1)}$

Solution: $x \to -1^-$ means $x < -1$, implying $x + 1 < 0$. Thus

$$\lim_{x\to -1^-} \frac{|1-x^2|}{x(x+1)} = \lim_{x\to -1^-} \frac{|(1-x)(1+x)|}{x(x+1)} = \lim_{x\to -1^-} \frac{(1-x)(-(1+x))}{x(x+1)} = \lim_{x\to -1^-} \frac{x-1}{x} = 2$$

(c) [4 marks] $\lim_{x\to -\infty} \frac{\sqrt{x^2+2x-5}-3x}{x-6}$

Solution: divide numerator and denominator by x :

$$\begin{aligned}
\lim_{x\to -\infty} \frac{\sqrt{x^2+2x-5}-3x}{x-6} &= \lim_{x\to -\infty} \frac{\sqrt{x^2+2x-5}/x-3}{1-6/x} \\
(\text{since } x<0,\ x=-\sqrt{x^2}) &= \lim_{x\to -\infty} \frac{-\sqrt{\frac{x^2+2x-5}{x^2}}-3}{1-6/x} \\
&= \lim_{x\to -\infty} \frac{-\sqrt{1+2/x-5/x^2}-3}{1-6/x} \\
&= -\frac{4}{1} = -4
\end{aligned}$$

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