MAT186H1F - Calculus I - Fall 2016

Solutions to Term Test 1 - October 18, 2016

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

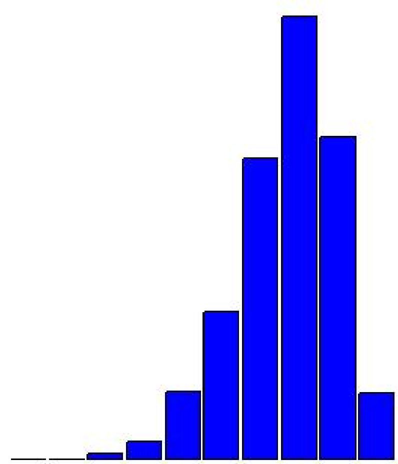
1. The range on each question was 0 (or 1 or 2) to 10; each question had a passing average.
2. Many students seem unaware of the *definitions* of the inverse trig functions, such as $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ and $\sec^{-1} x$; there are no choices:

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}, 0 \leq \cos^{-1} x \leq \pi, -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \text{ and } 0 \leq \sec^{-1} x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < \sec^{-1} x \leq \pi.$$

3.
4.

Breakdown of Results: 755 students wrote this test. The marks ranged from 23.75% to 100%, and the average was 71.0%. There was one perfect paper. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.8%
A	28.4%	80-89%	23.6%
B	32.3%	70-79%	32.3%
C	22.0%	60-69%	22.0%
D	10.7%	50-59%	10.7%
F	6.6%	40-49%	4.9%
		30-39%	1.3%
		20-29%	0.4%
		10-19%	0.0%
		0-9%	0.0%



1. [avg: 6.9/10] Indicate if the following statements are True or False. No justification is required; 1 mark for each correct choice.

(a) A tangent line is the limit of secant lines. ⊗ **True** ○ **False**

(b) A definite integral is the limit of Riemann sums. ⊗ **True** ○ **False**

(c) If $\lim_{x\to\infty} f'(x) = 0$ then $\lim_{x\to\infty} f(x) \neq \infty$ ○ **True** ⊗ **False**

(d) $\sec^{-1}\sqrt{2} = -\frac{\pi}{4}$ ○ **True** ⊗ **False**

(e) $\lim_{x\to\infty} \frac{1+2\cos x}{x} = 0$ ⊗ **True** ○ **False**

(f) If $f(x)$ is continuous on the interval $[a, b]$ and $F'(x) = f(x)$ for all x in the interval $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$. ⊗ **True** ○ **False**

(g) The definite integral of f on the interval $[a, b]$ represents the area of the region under the graph of f on $[a, b]$. ○ **True** ⊗ **False**

(h) $\sin(\sin^{-1} x) = x$ for all x in $[-1, 1]$. ⊗ **True** ○ **False**

(i) $\cos^{-1}(\cos\theta) = \theta$, for all θ. ○ **True** ⊗ **False**

(j) If $x > 0$ then $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$ ⊗ **True** ○ **False**

2. [avg: 6.7/10] Suppose the function $g(t)$ is defined for all t in the interval $[0, 8]$; the only discontinuities of g in the interval $[0, 8]$ are at $t = 2.5, 4$ and 8; and that the following data points are on the graph of g :

t	0	1	2	3	4	5	6	7	8
$g(t)$	-1.1	5.7	2.2	-9.0	0.2	-0.3	-4.6	8.8	-3.2

(a) [2 marks] What is the slope of the secant line joining the points $(6, g(6))$ and $(8, g(8))$?

Solution: let m be the slope of the secant line.

$$m = \frac{g(8) - g(6)}{8 - 6} = \frac{-3.2 + 4.6}{2} = 0.7$$

(b) [3 marks] Approximate the value of $\int_0^8 g(t)\,dt$ by using a left Riemann sum and $n = 4$ subintervals of $[0, 8]$.

Solution: use a regular partition with $\Delta x = 2$:

$$\int_0^8 g(t)\,dt \approx g(0)\,\Delta x + g(2)\,\Delta x + g(4)\,\Delta x + g(6)\,\Delta x = 2(-1.1 + 2.2 + 0.2 - 4.6) = -6.6$$

(c) [3 marks] At least how many solutions in the interval $[0, 8]$ are there to the equation $g(t) = 0$?

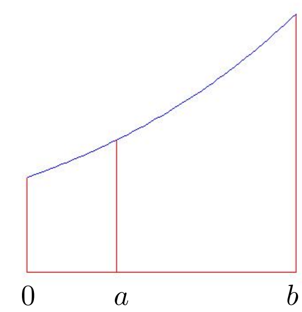
Solution: g is not continuos on the intervals $[2, 3], [3, 4], [4, 5]$ or $[7, 8]$; g is continuos on the intervals $[0, 1], [1, 2], [5, 6]$ and $[6, 7]$. On only two of these, namely $[0, 1]$ and $[6, 7]$, does g change signs. Thus by the Intermediate Theorem there will be at least one solution to $g(t) = 0$ in the interval $[0, 1]$ and at least one solution to $g(t) = 0$ in the interval $[6, 7]$. Answer: at least two.

(d) [2 marks] At least how many points are there in the interval $(0, 8)$ at which g is not differentiable?

Solution: we know that if g is differentiable at $x = a$ then it is continuous at $x = a$. This means: if g is not continuous at $x = a$ it is not differentiable at $x = a$. Thus there are at least two points in the interval $(0, 8)$ at which g is not differentiable: $x = 2.5$ and $x = 4$.

3. [avg: 7.0/10]

3.(a) [5 marks] Let A be the area of the region between $y = e^x$ and $y = 0$ on the interval $[0, a]$; let B be the area of the region between $y = e^x$ and $y = 0$ on the interval $[0, b]$. See the diagram to the right. Suppose B is four times A. Express b in terms of a.



Solution: we have

$$A = \int_0^a e^x \, dx = [e^x]_0^a = e^a - 1$$

and

$$B = \int_0^b e^x \, dx = [e^x]_0^b = e^b - 1.$$

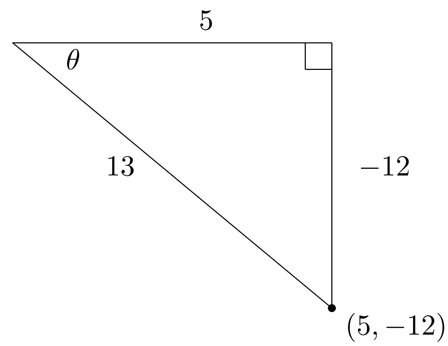
If $B = 4A$, then

$$e^b - 1 = 4(e^a - 1) \Rightarrow e^b = 4e^a - 3 \Rightarrow b = \ln(4e^a - 3).$$

3.(b) [5 marks] Find the exact value of $\sin\left(2\tan^{-1}\left(-\frac{12}{5}\right)\right)$

Solution: let

$$\theta = \tan^{-1}\left(-\frac{12}{5}\right).$$



Then $-\pi/2 < \theta < 0$ and

$$\tan\theta = -\frac{12}{5}$$

and

$$\sin\theta = -\frac{12}{13}, \ \cos\theta = \frac{5}{13}.$$

Finally,

$$\sin(2\theta) = 2\sin\theta\,\cos\theta = 2\left(-\frac{12}{13}\right)\left(\frac{5}{13}\right) = -\frac{120}{169}.$$

4. [avg: 9.2/10] Let

$$f(x) = \frac{1+x}{x^2\,e^x}.$$

Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(1, f(1))$.

Solution: use quotient and product rules, as required:

$$f'(x) = \frac{1 \cdot x^2 e^x - (1+x) \cdot (2xe^x + x^2 e^x)}{x^4 e^{2x}} = -\frac{2 + 2x + x^2}{x^3\,e^x}.$$

Then

$$f'(1) = -\frac{5}{e}.$$

The equation of the tangent line to f at $x = 1$ is given by

$$\frac{y - f(1)}{x - 1} = f'(1) \Leftrightarrow \frac{y - 2/e}{x - 1} = -\frac{5}{e} \Leftrightarrow y = -\frac{5}{e}x + \frac{7}{e}.$$

5. [avg: 7.2/10] Find the following limits.

(a) [5 marks] $\displaystyle\lim_{x\to 2} \frac{\sqrt{x^2+21}-5}{x^2-x-2}$

Solution: rationalize and factor:

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x^2+21}-5}{x^2-x-2} &= \lim_{x\to 2} \frac{(\sqrt{x^2+21}-5)(\sqrt{x^2+21}+5)}{(x^2-x-2)(\sqrt{x^2+21}+5)} \\
&= \lim_{x\to 2} \frac{x^2-4}{(x-2)(x+1)(\sqrt{x^2+21}+5)} \\
&= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+1)(\sqrt{x^2+21}+5)} \\
&= \lim_{x\to 2} \frac{x+2}{(x+1)(\sqrt{x^2+21}+5)} \\
&= \frac{2}{15}
\end{aligned}$$

(b) [5 marks] $\displaystyle\lim_{x\to 0} \frac{\tan(3x)}{\sin(5x)}$

Solution: manipulate the expression until you can use the basic trig limit

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(3x)}{\sin(5x)} &= \lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} \cdot \lim_{h\to 0} \frac{1}{\cos(3x)} \\
&= \frac{3}{5} \cdot \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \lim_{x\to 0} \frac{5x}{\sin(5x)} \cdot 1 \\
&= \frac{3}{5} \cdot 1 \cdot \frac{1}{1}, \text{ using basic trig limit, twice} \\
&= \frac{3}{5}
\end{aligned}$$

6. [avg: 6.1/10] Find all the asymptotes—vertical, horizontal, or slant—of the following functions:

(a) [5 marks] $p(x) = \dfrac{x^3 - 3x^2 + 2x}{x^2 - 1}$

Solution: factor, simplify and divide:

$$p(x) = \frac{x^3 - 3x^2 + 2x}{x^2 - 1} = \frac{(x^2 - 2x)(x-1)}{(x-1)(x+1)} = \frac{x^2 - 2x}{x+1} = x - 3 + \frac{3}{x+1},$$

if $x \neq 1$. **In particular,** there is no vertical asymptote at $x = 1$ since

$$\lim_{x \to 1} p(x) = -\frac{1}{2}.$$

The line with equation $y = x - 3$ is a slant asymptote, and the line $x = -1$ is a vertical asymptote, since

$$\lim_{x \to -1^-} p(x) = -\infty \text{ and } \lim_{x \to -1^+} p(x) = \infty$$

(b) [5 marks] $q(x) = \dfrac{1}{1 - e^x}$

Solution: $q(0)$ is not defined, because $e^0 = 1$. There is a vertical asymptote at $x = 0$ since

$$\lim_{x \to 0^-} q(x) = \infty \text{ and } \lim_{x \to 0^+} q(x) = -\infty.$$

As for horizontal asymptotes, they exist on both sides:

$$\lim_{x \to -\infty} q(x) = \frac{1}{1-0} = 1, \text{ since } \lim_{x \to -\infty} e^x = 0,$$

and

$$\lim_{x \to \infty} q(x) = 0, \text{ since } \lim_{x \to \infty} e^x = \infty.$$

Thus there are horizontal asymptotes at $y = 0$ and $y = 1$.

7. [avg: 6.7/10] Suppose the position at time t of an object moving horizontally along the x-axis is given by $x = t^2 - 4t$, for $0 \le t \le 8$.

(a) [2 marks] What is the velocity of the object at time t?

Solution: let the velocity of the object be v.

$$v = \frac{dx}{dt} = 2t - 4.$$

(b) [2 marks] For which values of t is the object moving towards the left?

Solution: $v < 0 \Leftrightarrow 2t - 4 < 0 \Leftrightarrow t < 2$. So the object is moving to the left when $0 \le t < 2$.

(c) [2 marks] What is the average velocity of the object on the time interval $0 \le t \le 3$?

Solution: the average velocity is

$$\frac{\Delta x}{\Delta t} = \frac{x(3) - x(0)}{3 - 0} = \frac{-3 - 0}{3} = -1$$

(d) [2 marks] What is the speed of the object at $t = 1$?

Solution: speed is $|v|$. At $t = 1, v = -2$, so the speed at $t = 1$ is 2.

(e) [2 marks] What is the average speed of the object on the time interval $0 \le t \le 3$?

Solution: the average speed is the total distance travelled divided by the total time. In the first 2 seconds the object goes from $x = 0$ to $x = -4$, so the object travels 4 units. In the next second the object goes form $x = -4$ to $x = -3$, a further 1 unit distance travelled. So the total distance travelled in the first 3 seconds is 5; and the average speed is $\frac{5}{3}$.

Alternate Solution: or use integrals and the fact that the total distance travelled by the object over the first 3 seconds is

$$\int_0^3 |v|\,dt = \int_0^2 -(2t - 4)\,dt + \int_2^3 (2t - 4)\,dt = \left[4t - t^2\right]_0^2 + \left[t^2 - 4t\right]_2^3 = 4 + 1 = 5$$

so the average speed is 5/3, as before.

8. [avg: 7.0/10] The displacement of a mass on a spring suspended from the ceiling is given by

$$y = 10\,e^{-t/2}\cos\left(\frac{\pi\,t}{8}\right).$$

(a) [5 marks] Find the velocity of the mass at time t.

Solution: let v be the velocity of the mass.

$$v = \frac{dy}{dt} = -5e^{-t/2}\cos\left(\frac{\pi t}{8}\right) - \frac{5\pi}{4}e^{-t/2}\sin\left(\frac{\pi\,t}{8}\right) = -5e^{-t/2}\left(\cos\left(\frac{\pi t}{8}\right) + \frac{\pi}{4}\sin\left(\frac{\pi\,t}{8}\right)\right).$$

(b) [3 marks] At which time, $t > 0$, is the velocity first equal to zero?

Solution: set $v = 0$:

$$\begin{aligned}
-5e^{-t/2}\left(\cos\left(\frac{\pi t}{8}\right) + \frac{\pi}{4}\sin\left(\frac{\pi\,t}{8}\right)\right) = 0 \quad &\Rightarrow \quad \cos\left(\frac{\pi t}{8}\right) + \frac{\pi}{4}\sin\left(\frac{\pi\,t}{8}\right) = 0\\
&\Rightarrow \quad \frac{\pi}{4}\sin\left(\frac{\pi\,t}{8}\right) = -\cos\left(\frac{\pi t}{8}\right)\\
&\Rightarrow \quad \tan\left(\frac{\pi\,t}{8}\right) = -\frac{4}{\pi}\\
&\Rightarrow \quad \frac{\pi\,t}{8} = \tan^{-1}\left(-\frac{4}{\pi}\right) = -\tan^{-1}\left(\frac{4}{\pi}\right)\\
&\Rightarrow \quad t = -\frac{8}{\pi}\tan^{-1}\left(\frac{4}{\pi}\right) < 0,
\end{aligned}$$

so, since the period of $\tan(\pi t/8)$ is 8, take

$$t = 8 - \frac{8}{\pi}\tan^{-1}\left(\frac{4}{\pi}\right) \approx 5.7\ldots.$$

Note: answer must be in radians!

(c) [2 marks] What is $\lim_{t\to\infty} y$?

Solution: use the squeeze law, and the fact that

$$-10 \le 10\cos\left(\frac{\pi\,t}{8}\right) \le 10.$$

Thus

$$-10e^{-t/2} \le 10e^{-t/2}\cos\left(\frac{\pi\,t}{8}\right) \le 10e^{-t/2}.$$

Since

$$\lim_{t\to\infty} e^{-t/2} = 0,$$

the expression squeezed in the middle also has limit zero: $\lim_{t\to\infty} y = 0$.

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