Solutions to MAT186H1F - Calculus I - Fall 2014

Term Test 1 - October 7, 2014

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

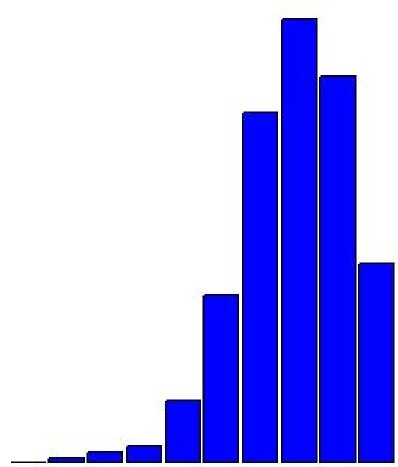
This test consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. Questions 1, 3, 5 and 6 had the best results; Questions 2 and 8 had the worst results.
2. In Q2 you had to explain why the correct choice in parts (a) and (c) was the positive square root.
3. In Q6(b) the instantaneous velocity is $400/121 \approx 3.305$; it is incorrect mathematically to say that $400/121 = 3.305$.
4. In Q8(a), you must state the Intermediate Value Theorem as given in the book.

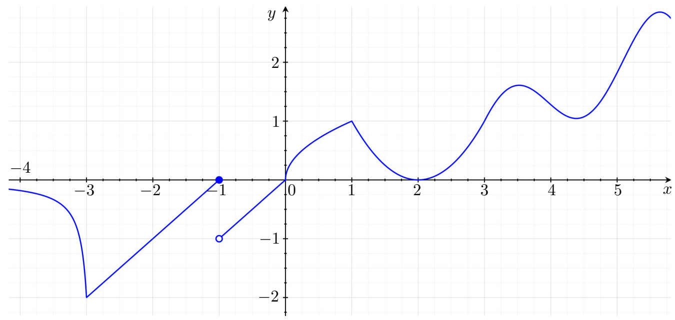
Breakdown of Results: 908 students wrote this test. The marks ranged from 10% to 100%, and the average was 73%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	12.1%
A	35.7%	80-89%	23.6%
B	27.1%	70-79%	27.1%
C	21.4%	60-69%	21.4%
D	10.2%	50-59%	10.2%
F	5.6%	40-49%	3.7%
		30-39%	1.0%
		20-29%	0.7%
		10-19%	0.2%
		0-9%	0.0%



PART I : No explanation is necessary.

1. (avg: 8.2) The graph of the function f is given to the right. Decide if the following statements about f are True or False. **Circle your answers.**



(a) $f'(-2) = 1$ True

(b) f is one-to-one for $0 \leq x \leq 5$ False

(c) f is differentiable at $x = -3$ False

(d) f is continuous at $x = -3$ True

(e) f is differentiable at $x = -1$ False

(f) f is continuous at $x = -1$ False

(g) $\lim_{x \to -3^-} f'(x) < \lim_{x \to 4^-} f'(x)$ True

(h) $|f'(4)| < |f'(3)|$ True

(i) There are exactly 2 solutions to the equation $f(x) = \dfrac{1}{2}$. False

(j) The function $f'(x)$ has four discontinuities. True

PART II : Present **complete** solutions to the following questions in the space provided.

2. (avg: 5.9) Given that $\theta = \cos^{-1}\left(-\frac{2}{3}\right)$ find the *exact* values of the following:

   (a) [3 marks] $\sin\theta$

   Solution: by definition of inverse cosine, $\cos\theta = -2/3$ and $0 \le \theta \le \pi$. So θ must be in the second quadrant, that is $\pi/2 < \theta < \pi$, and consequently

$$\sin\theta = +\sqrt{1-\cos^2\theta)} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}.$$

   (b) [3 marks] $\cos(2\theta)$

   Solution:

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{4}{9}\right) - 1 = -\frac{1}{9}.$$

   (c) [4 marks] $\sin(\theta/2)$

   Solution: use

$$\sin^2(\theta/2) = \frac{1-\cos\theta}{2} = \frac{1+2/3}{2} = \frac{5}{6}.$$

   Since

$$\frac{\pi}{2} < \theta < \pi \Rightarrow \frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2} \Rightarrow \sin\left(\frac{\theta}{2}\right) > 0,$$

   the answer is

$$\sin(\theta/2) = +\sqrt{\frac{5}{6}}.$$

3. (avg: 8.5) Find the following limits.

(a) [3 marks] $\lim_{x\to 2^+}\left(\frac{3}{x-2}-\frac{6}{x^2-4}\right)$

Solution:

$$\lim_{x\to 2^+}\left(\frac{3}{x-2}-\frac{6}{x^2-4}\right)=\lim_{x\to 2^+}\frac{3(x+2)-6}{x^2-4}=\lim_{x\to 2^+}\frac{3x}{x^2-4}=\infty$$

(b) [3 marks] $\lim_{x\to 4^-}\frac{|x^2-16|}{x-4}$.

Soluton: since $x<4$ and close to 4, we have $x^2<16$ and $|x^2-16|=-(x^2-16)$; thus

$$\lim_{x\to 4^-}\frac{|x^2-16|}{x-4}=\lim_{x\to 4^-}\frac{-(x^2-16)}{x-4}=-\lim_{x\to 4^-}\frac{(x-4)(x+4}{x-4}=-\lim_{x\to 4^-}(x+4)=-8.$$

(c) [2 marks] $\lim_{x\to 0}\frac{\sin(4x)}{3x}$.

Solution: make use of the basic trig limit.

$$\lim_{x\to 0}\frac{\sin(4x)}{3x}=\frac{4}{3}\lim_{x\to 0}\frac{\sin(4x)}{4x}=\frac{4}{3}\lim_{h\to 0}\frac{\sin(h)}{h}=\frac{4}{3}\cdot 1=\frac{4}{3}.$$

(d) [2 marks] $\lim_{x\to\infty}\frac{\sqrt{4x^2+9x}}{3x}$.

Solution:

$$\lim_{x\to\infty}\frac{\sqrt{4x^2+9x}}{3x}=\frac{1}{3}\sqrt{\lim_{x\to\infty}\frac{4x^2+9x}{x^2}}=\frac{1}{3}\sqrt{\lim_{x\to\infty}\left(4+\frac{9}{x}\right)}=\frac{2}{3}.$$

4. (avg: 7.5) The parts of this question are unrelated.

(a) [4 marks] Find $\lim_{x\to 1} \dfrac{\sqrt{3x+1}-\sqrt{5x-1}}{x-1}$

Solution: rationalize.

$$\lim_{x\to 1} \frac{\sqrt{3x+1}-\sqrt{5x-1}}{x-1} = \lim_{x\to 1} \frac{\sqrt{3x+1}-\sqrt{5x-1}}{x-1} \cdot \frac{\sqrt{3x+1}+\sqrt{5x-1}}{\sqrt{3x+1}+\sqrt{5x-1}}$$

$$= \lim_{x\to 1} \frac{3x+1-(5x-1)}{(x-1)(\sqrt{3x+1}+\sqrt{5x-1})} = \lim_{x\to 1} \frac{-2(x-1)}{(x-1)(\sqrt{3x+1}+\sqrt{5x-1})}$$

$$= \lim_{x\to 1} \frac{-2}{(\sqrt{3x+1}+\sqrt{5x-1})} = -\frac{2}{2+2} = -\frac{1}{2}$$

(b) [6 marks] Find all vertical, horizontal, and slant asymptotes of the function $f(x) = \dfrac{x^3-4x}{x^2+x-6}$.

Solution: Since the degree of the numerator is greater than the degree of the denominator, there are no **horizontal** asymptotes.

For the rest: factor and divide, for $x \neq 2$:

$$\frac{x^3-4x}{x^2+x-6} = \frac{x(x^2-4)}{(x-2)(x+3)} = \frac{x(x+2))}{(x+3)} = \frac{x^2+2x}{x+3} = x-1+\frac{3}{x+3}.$$

So $y = x-1$ is a **slant** asymptote of f and $x = -3$ is a **vertical** asymptote of f.

Note: $x = 2$ is **not** a vertical asymptote; $x = 2$ is a removable discontinuity of f.

5. (avg: 8.6) Find and simplify $\dfrac{dy}{dx}$ if

(a) [5 marks] $y = \dfrac{\cos(e^x)}{1+\sin(e^x)}$.

Solution: use the quotient rule and the chain rule.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-e^x\sin(e^x)(1+\sin(e^x)) - \cos(e^x)(e^x\cos(e^x))}{(1+\sin(e^x))^2} \\
&= \frac{-e^x\sin(e^x) - e^x\sin^2(e^x) - e^x\cos^2(e^x))}{(1+\sin(e^x))^2} \\
&= \frac{-e^x\sin(e^x) - e^x}{(1+\sin(e^x))^2} = -\frac{e^x}{1+\sin(e^x)}
\end{aligned}$$

(b) [5 marks] $y = \ln(x + \sqrt{1+x^2})$.

Solution: use the chain rule.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right) \\
&= \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{\left(\sqrt{1+x^2}+x\right)}{\sqrt{1+x^2}} \\
&= \frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

6. (avg: 8.4) The position (in meters) of a marble rolling up a long incline is given by $s = \dfrac{100t}{t+1}$, where t is measured in seconds and $s = 0$ is the starting point.

(a) [3 marks] Find the average velocity of the marble over the first 9 seconds, $0 \le t \le 9$.

Solution: average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{\frac{100(9)}{10} - 0}{9} = 10$$

(b) [4 marks] Find the instantaneous velocity of the marble at $t = 4.5$ sec.

Solution: $s = \dfrac{100t}{t+1} = 100 - \dfrac{100}{t+1}$, so

$$v = \frac{ds}{dt} = \frac{100}{(t+1)^2}.$$

At $t = 4.5$ the instantaneous velocity is

$$\frac{100}{5.5^2} = \frac{400}{121}.$$

(c) [3 marks] Find both $\lim\limits_{t\to\infty} s$ and $\lim\limits_{t\to\infty} v$, where v is the velocity of the particle at time t.

Solution:

$$\lim_{t\to\infty} s = \lim_{t\to\infty} \left(100 - \frac{100}{t+1}\right) = 100$$

and

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{100}{(t+1)^2} = 0.$$

7. (avg: 6.7) Let $f(x) = x^2 + x$. Find all tangent lines to the graph of f that pass through the point $(x, y) = (2, -3)$, and illustrate your solution with a diagram.

Solution: let the point of contact of the tangent line from $(2, -3)$ be $(a, f(a))$; the equation of the tangent line is

$$y = f(a) + f'(a)(x - a) = a^2 + a + (2a + 1)(x - a).$$

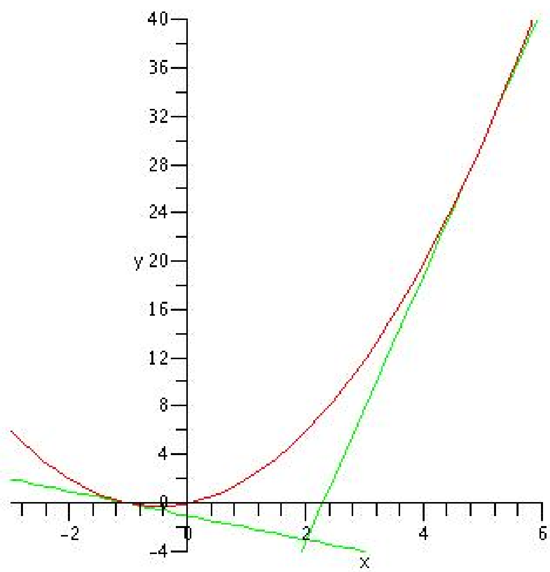
If this tangent line is to pass through the point $(x, y) = (2, -3)$, then

$$\begin{aligned} -3 = a^2 + a + (2a+1)(2-a) \quad &\Leftrightarrow \quad -3 = a^2 + a + 4a + 2 - 2a^2 - a \\ &\Leftrightarrow \quad a^2 - 4a - 5 = 0 \\ &\Leftrightarrow \quad (a-5)(a+1) = 0 \end{aligned}$$

Thus there are two tangent lines to the graph of f that pass through the point $(2, -3)$, for $a = 5$ and $a = -1$:

$$y = 30 + 11(x - 5) = -25 + 11x \text{ and } y = 0 - (x + 1) = -x - 1.$$

Diagram:



8.(avg: 4.7)

(a) [4 marks] State the Intermediate Value Theorem.

Intermediate Value Theorem: Let f be a continuous function on the closed interval $[a, b]$ and suppose K is a value between $f(a)$ and $f(b)$. The there is a number c in the open interval (a, b) such that $f(c) = K$.

8.(b) [6 marks] Use the Intermediate Value Theorem to explain, clearly and concisely, why the equation $x^3 + x^2 - 2x = 1$ has at least two solutions in the interval $[-2, 0]$.

Solution: Let $f(x) = x^3 + x^2 - 2x$, which is a polynomial function, so is continuous everywhere. We have

$$f(-2) = 0, \quad f(-1) = 2 \text{ and } f(0) = 0.$$

On the interval $[-2, -1]$,

$$f(-2) = 0 < 1 < 2 = f(-1)$$

so by IVT there is a number c in $(-2, -1)$, such that $f(c) = 1$.

Similarly on the interval $[-1, 0]$,

$$f(-1) = 2 > 1 > f(0)$$

so by IVT there is a number d in $(-1, 0)$, such that $f(d) = 1$.

Thus there are at least two solutions to the equation $f(x) = 1$ in the interval $[-2, 0]$.

This page is for rough work; it will not be marked.