University of Toronto
Solutions to **MAT186H1F TERM TEST**
of **Tuesday, October 15, 2013**
Duration: 100 minutes

Only aids permitted: Casio FX-991 or Sharp EL-520 calculator.

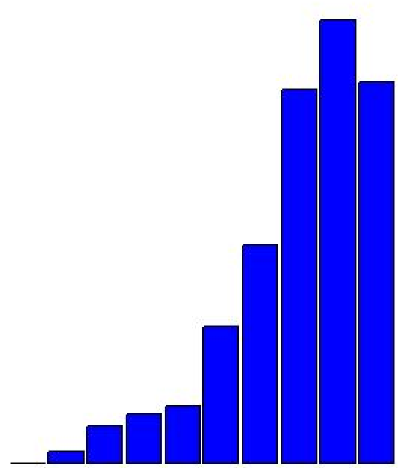
Instructions: Answer all questions. Present your solutions in the space provided; use the backs of the pages if you need more space. Do not use L'Hopital's rule on this test. The value for each question is indicated in parantheses beside the question number. **Total Marks: 60**

General Comments about the Test:

- In a written test you must explain what you are doing to get full credit. The answer by itself is worth very little if you don't explain how you got it. Moreover, you can't just plop down formulas and expect the marker to figure out what you are doing; you are supposed to make it clear what you are doing.
- To get full marks in 2(a) you must explain why $\cos\theta > 0$.
- In 5(a) many students wrote what looked like $\sin(x^2) = x \sin x$, which is not true. Moreover, you must refer to the basic trig limit,
$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$
and make it clear how it applies to your solution, to get full marks.
- Most of the problems on this test were based on WeBWorK questions, or exercises in the book. None of them should have come as any surprise. (And it looks like they didn't.)

Breakdown of Results: 448 students wrote this test. The marks ranged from 10% to 100%, and the average was 74.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	22.3%
A	48.2%	80-89%	25.9%
B	21.9%	70-79%	21.9%
C	12.7%	60-69%	12.7%
D	8.1%	50-59%	8.1%
F	9.1%	40-49%	3.3%
		30-39%	2.9%
		20-29%	2.2%
		10-19%	0.7%
		0-9%	0.0%



1. [8 marks; 4 marks for each part.] Find and simplify $\dfrac{dy}{dx}$ if

(a) $y = \dfrac{\tan x}{1 + x\tan x}$.

Solution: use the quotient rule, and product rule.

$$\frac{dy}{dx} = \frac{\sec^2 x\,(1 + x\tan x) - \tan x\,(\tan x + x\sec^2 x)}{(1 + x\tan x)^2} = \frac{1}{(1 + x\tan x)^2}$$

(b) $y = \ln\sqrt{\dfrac{4x-3}{3x+9}}$, for $x > 3/4$.

Solution 1: use properties of logarithms.

$$y = \ln\left(\frac{4x-3}{3x+9}\right)^{1/2} = \frac{1}{2}\left(\ln(4x-3) - \ln(3x+9)\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{4}{4x-3} - \frac{3}{3x+9}\right) = \frac{1}{2}\frac{15}{(4x-3)(x+3)} \text{ or } \frac{1}{2}\frac{15}{(4x^2+9x-9)}$$

Solution 2: use chain rule and quotient rule.

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4x-3}{3x+9}}}\frac{1}{2\sqrt{\frac{4x-3}{3x+9}}}\frac{4(3x+9) - 3(4x-3)}{(3x+9)^2} = \frac{1}{2}\frac{15}{(4x-3)(x+3)} \text{ or } \frac{1}{2}\frac{15}{(4x^2+9x-9)}$$

Solution 3: use properties of exponentials and implicit differentiation.

$$e^{2y} = \frac{4x-3}{3x+9} \Rightarrow 2e^{2y}\frac{dy}{dx} = \frac{4(3x+9) - 3(4x-3)}{(3x+9)^2} = \frac{45}{(3x+9)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2e^{2y}}\frac{45}{(3x+9)^2} = \frac{1}{2}\frac{3x+9}{4x-3}\frac{45}{(3x+9)^2} = \frac{1}{2}\frac{15}{(4x-3)(x+3)} \text{ or } \frac{1}{2}\frac{15}{(4x^2+9x-9)}$$

2. [7 marks] Given that $\theta = \sin^{-1}\left(-\frac{2}{3}\right)$ find the *exact* values of the following:

(a) [3 marks] $\cos\theta$

Solution: by definition of $\sin^{-1}$,

$$\sin\theta = -\frac{2}{3} \text{ and } -\frac{\pi}{2} < \theta < 0.$$

So $\cos\theta > 0$ and

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(-\frac{2}{3}\right)^2} = \sqrt{\frac{5}{9}} \text{ or } \frac{\sqrt{5}}{3}.$$

(b) [4 marks] $\cos(2\theta)$

Solution: use the appropriate double angle formula.

$$\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(-\frac{2}{3}\right)^2 =, 1 - \frac{8}{9} = \frac{1}{9}$$

Alternately:

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{\sqrt{5}}{3}\right)^2 - 1 = \frac{10}{9} - 1 = \frac{1}{9}.$$

3. [7 marks] At what point does the normal line to the graph of $f(x) = -4 + 2x + 4x^2$ at the point $(1, 2)$ intersect the parabola a second time?

Solution:

$$f'(x) = 2 + 8x; f'(1) = 10,$$

so the equation of the normal line to the graph of f at $(1, 2)$ is

$$\frac{y-2}{x-1} = -\frac{1}{f'(1)} = -\frac{1}{10} \Leftrightarrow 10(y-2) = 1 - x \Leftrightarrow y = \frac{21-x}{10}.$$

Now find the intersection of this line and the graph of f:

$$\frac{21-x}{10} = -4 + 2x + 4x^2 \Leftrightarrow 21 - x = 40x^2 + 20x - 40 \Leftrightarrow 0 = 40x^2 + 21x - 61.$$

Factoring:

$$40x^2 + 21x - 61 = (x-1)(40x+61).$$

Thus

$$40x^2 + 21x - 61 = 0 \Leftrightarrow x = 1 \text{ or } x = -\frac{61}{40}$$

and the normal line to f at $(1, 2)$ intersects the graph of f a second time at

$$(x, y) = \left(-\frac{61}{40}, f\left(-\frac{61}{40}\right)\right) = \left(-\frac{61}{40}, \frac{901}{400}\right).$$

4. [8 marks; 4 marks for each part.] Find the following limits:

(a) $\lim\limits_{x\to 2} \dfrac{\sqrt{3x+10}-\sqrt{4x+8}}{x-2}$

Solution: the limit is in the 0/0 form; rationalize the numerator.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{3x+10}-\sqrt{4x+8}}{x-2} &= \lim_{x\to 2} \frac{(\sqrt{3x+10}-\sqrt{4x+8})(\sqrt{3x+10}+\sqrt{4x+8})}{(x-2)(\sqrt{3x+10}+\sqrt{4x+8})} \\
&= \lim_{x\to 2} \frac{3x+10-(4x+8)}{(x-2)(\sqrt{3x+10}+\sqrt{4x+8})} \\
&= \lim_{x\to 2} \frac{2-x}{(x-2)(\sqrt{3x+10}+\sqrt{4x+8})} \\
&= -\lim_{x\to 2} \frac{1}{\sqrt{3x+10}+\sqrt{4x+8}} \\
&= -\frac{1}{\sqrt{16}+\sqrt{16}} = -\frac{1}{8}
\end{aligned}$$

(b) $\lim\limits_{x\to 1} \left(\dfrac{3}{x-1} - \dfrac{6}{x^2-1}\right)$

Solution: the limit is in the $\infty - \infty$ form; get a common denominator.

$$\begin{aligned}
\lim_{x\to 1} \left(\frac{3}{x-1} - \frac{6}{x^2-1}\right) &= \lim_{x\to 1} \frac{3(x+1)-6}{x^2-1} \\
&= \lim_{x\to 1} \frac{3(x-1)}{(x-1)(x+1)} \\
&= \lim_{x\to 1} \frac{3}{(x+1)} \\
&= \frac{3}{2}
\end{aligned}$$

5. [8 marks; 4 marks for each part.] Find the following limits:

(a) $\lim_{x\to 0} \frac{\sin(x^2)}{x}$.

Solution: this limit is in the 0/0 indeterminate form. To reduce this limit to an application of the basic trig limit,

$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$

let $t = x^2$:

$$\lim_{x\to 0} \frac{\sin(x^2)}{x} = \lim_{x\to 0} x \left(\frac{\sin(x^2)}{x^2}\right) = \left(\lim_{x\to 0} x\right)\left(\lim_{t\to 0} \frac{\sin t}{t}\right) = 0 \times 1 = 0$$

(b) $\lim_{x\to -\infty} \frac{\sqrt{x^2+11x}}{8-10x}$

Solution: this limit is in the ∞/∞ indeterminate form. Factor inside the square root sign.

$$\begin{aligned}
\lim_{x\to -\infty} \frac{\sqrt{x^2+11x}}{8-10x} &= \lim_{x\to -\infty} \frac{\sqrt{x^2(1+11/x)}}{8-10x} \\
&= \lim_{x\to -\infty} \frac{\sqrt{x^2}\,\sqrt{1+11/x}}{8-10x} \\
&= \lim_{x\to -\infty} \frac{|x|\,\sqrt{1+11/x}}{8-10x} \\
\text{(since } x<0\text{)} \quad &= \lim_{x\to -\infty} \frac{-x\,\sqrt{1+11/x}}{8-10x} \\
&= \lim_{x\to -\infty} \frac{-\sqrt{1+11/x}}{8/x-10} \\
&= \frac{-\sqrt{1+0}}{0-10} \\
&= \frac{1}{10}
\end{aligned}$$

6. [7 marks] Find the values of a and b so that

$$f(x) = \begin{cases} ax^2 + 2x - 1 & \text{if} \quad x \le -2, \\ x^2 + bx & \text{if} \quad x > -2 \end{cases}$$

is differentiable for all x.

Solution: differentiability implies continuity, so one condition that a and b must satisfy is

$$\lim_{x \to -2} f(x) = f(-2) \Rightarrow \lim_{x \to -2^+} (x^2 + bx) = f(-2) \Rightarrow 4 - 2b = 4a - 5 \Rightarrow 4a + 2b = 9. \quad (1)$$

Since f is to be differentiable at $x = -2$ another condition that a and b must satisfy is

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h} &= \lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h} \\ \text{or equivalently} \quad \lim_{x \to -2^-} f'(x) &= \lim_{x \to -2^+} f'(x) \\ \Rightarrow \lim_{x \to -2^-} (2ax + 2)) &= \lim_{x \to -2^+} (2x + b) \\ \Rightarrow -4a + 2 &= -4 + b \\ \Rightarrow 4a + b &= 6. \quad (2) \end{aligned}$$

Solving the system of equations (1) and (2) gives

$$a = \frac{3}{4} \text{ and } b = 3.$$

7.(a) [3 marks] State the Intermediate Value Theorem.

Solution: quoting Theorem 1.5.7 from page 115 of the text book:

> If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$ then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.

7.(b) [4 marks] Use the Intermediate Value Theorem to explain, clearly and concisely, why the equation $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[1, 2]$.

Solution: let $f(x) = x^3 + x^2 - 2x$, which is a polynomial function and so is continuous for all x. Now consider $f(x)$ on the closed interval $[1, 2]$:

$$f(1) = 0 < 1 \text{ and } f(2) = 8 > 1.$$

Let $k = 1$. By the Intermediate Value Theorem there is a number $x \in [1, 2]$ such that

$$f(x) = k \Leftrightarrow x^3 + x^2 - 2x = 1.$$

8. [8 marks] Find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(x, y) = (1, -2/5)$ if $\frac{y}{x+2y} = x^9 - 3$.

Solution 1: cross multiply and differentiate implicitly.

$$\frac{y}{x+2y} = x^9 - 3 \Rightarrow y = (x^9-3)(x+2y) = x^{10} + 2x^9y - 3x - 6y$$
$$\Rightarrow \frac{dy}{dx} = 10x^9 + 18x^8y + 2x^9\frac{dy}{dx} - 3 - 6\frac{dy}{dx} \quad (1)$$

At $(x, y) = (1, -2/5)$, this becomes

$$\frac{dy}{dx} = 10 - \frac{36}{5} + 2\frac{dy}{dx} - 3 - 6\frac{dy}{dx} \Leftrightarrow 5\frac{dy}{dx} = -\frac{1}{5} \Leftrightarrow \frac{dy}{dx} = -\frac{1}{25} = -0.04.$$

To find $\frac{d^2y}{dx^2}$, differentiate equation (1) implicitly once more:

$$\frac{d^2y}{dx^2} = 90x^8 + 144x^7y + 18x^8\frac{dy}{dx} + 18x^8\frac{dy}{dx} + 2x^9\frac{d^2y}{dx^2} - 6\frac{d^2y}{dx^2}.$$

At $(x, y) = (1, -2/5)$, $\frac{dy}{dx} = -\frac{1}{25}$ so

$$\frac{d^2y}{dx^2} = 90 + 144\left(-\frac{2}{5}\right) + 18\left(-\frac{1}{25}\right) + 18\left(-\frac{1}{25}\right) + 2\frac{d^2y}{dx^2} - 6\frac{d^2y}{dx^2}$$
$$\Rightarrow 5\frac{d^2y}{dx^2} = \frac{2250 - 1440 - 36}{25} = \frac{774}{25} \Rightarrow \frac{d^2y}{dx^2} = \frac{774}{125} = 6.192.$$

Solution 2: cross multiply, solve for y, then differentiate explicitly.

$$\frac{y}{x+2y} = x^9 - 3 \Rightarrow y = (x^9-3)(x+2y) = x^{10} + 2x^9y - 3x - 6y \Rightarrow y = \frac{x^{10} - 3x}{7 - 2x^9}$$
$$\Rightarrow \frac{dy}{dx} = \frac{(10x^9-3)(7-2x^9) - (-18x^8)(x^{10}-3x)}{(7-2x^9)^2} = \frac{22x^9 - 2x^{18} - 21}{(7-2x^9)^2}$$

and after much work:

$$\frac{d^2y}{dx^2} = \frac{18x^8(35 + 8x^9)}{(7-2x^9)^3}.$$

Now substitute $x = 1$ to find, as before,

$$\frac{dy}{dx} = -\frac{1}{25} \text{ and } \frac{d^2y}{dx^2} = \frac{774}{125}.$$