University of Toronto
SOLUTIONS to MAT186H1F TERM TEST
of **Tuesday, October 18, 2011**
Duration: 110 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all questions. Present your solutions in the space provided; use the backs of the pages if you need more space. The value for each question is indicated in parantheses beside the question number. Do not use L'Hopital's rule on this test. TOTAL MARKS: 60

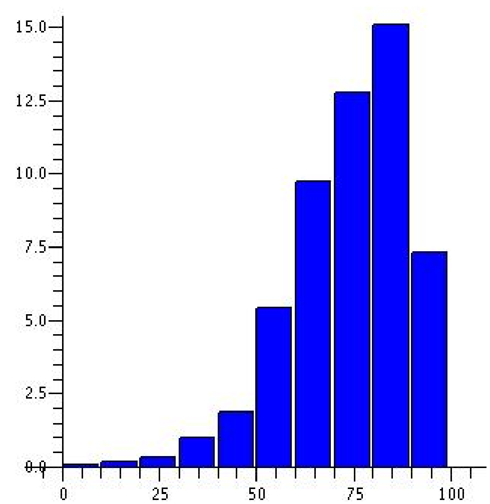
General Comments about the Test:

- In a written test you must explain what you are doing to get full credit. The answer by itself is worth very little if you don't explain how you got it. Moreover, you can't just plop down formulas and expect the marker to figure out what you are doing; you are supposed to make it clear what you are doing.
- In 3(a), $\pi/2 < \theta < \pi$, so $\sin\theta$ must be positive, not negative.
- In 5(a) you have to invoke the basic trig limit to get full marks.
- In 5(b), a lot of people wrote $\tan^{-1}(-\infty)$, which is meaningless. Indeed, there was lots of bad notation on far too many tests!
- In 6(b), for both $y = f(x)$ and $y = f^{-1}(x)$, $x < 0 \Leftrightarrow y < 0$. So for $3 < x \leq -1$,

$$f^{-1}(x) = -\sqrt{\frac{8x^2}{9-x^2}} = \frac{\sqrt{8}\,x}{\sqrt{9-x^2}}, \text{ since } x < 0.$$

Breakdown of Results: 486 students wrote this test. The marks ranged from 0% to 98.3%, and the average was 73.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	13.6%
A	41.6%	80-89%	28.0%
B	23.7%	70-79%	23.7%
C	18.1%	60-69%	18.1%
D	10.1%	50-59%	10.1%
F	6.6%	40-49%	3.5%
		30-39%	1.9%
		20-29%	0.6%
		10-19%	0.4%
		0-9%	0.2%



1. [8 marks; 4 for each part] Find $\frac{dy}{dx}$ if

(a) $y = \dfrac{\sin x}{\tan x + 1}$.

Solution: use the quotient rule.

$$\frac{dy}{dx} = \frac{\cos x(\tan x + 1) - \sec^2 x \, \sin x}{(\tan x + 1)^2} = \frac{\sin x + \cos x - \tan x \sec x}{(\tan x + 1)^2}$$

(b) $y = \sqrt{1 + \sqrt{x^2 + x^4}}$.

Solution: use the chain rule.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1 + \sqrt{x^2 + x^4}}} \cdot \frac{(2x + 4x^3)}{2\sqrt{x^2 + x^4}} = \frac{x + 2x^3}{2\sqrt{1 + \sqrt{x^2 + x^4}}\sqrt{x^2 + x^4}}$$

Note: $\sqrt{x^2 + x^4} = |x|\sqrt{1 + x^2}$, if you choose to make this simplification, so x is *not* a common factor that can be cancelled.

2. [6 marks] Find equations for two lines through the origin that are tangent to the ellipse with equation $2x^2 - 4x + y^2 + 1 = 0$.

Solution: differentiate implicitly to find dy/dx:

$$2x^2 - 4x + y^2 + 1 = 0 \Rightarrow 4x - 4 + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2-2x}{y}.$$

Let the point of contact of the tangent line and the ellipse be (a, b). Then (1)

$$2a^2 - 4a + b^2 + 1 = 0,$$

and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{2-2a}{b},$$

so the equation of the tangent line is

$$\frac{y-b}{x-a} = \frac{2-2a}{b}.$$

Substitute the point $(x, y) = (0, 0)$ into the equation of tangent line to obtain (2)

$$\frac{-b}{-a} = \frac{2-2a}{b} \Leftrightarrow b^2 = 2a - 2a^2 \Leftrightarrow 2a^2 - 2a + b^2 = 0.$$

Subtracting equation (2) from equation (1) gives

$$-2a + 1 = 0 \Leftrightarrow a = \frac{1}{2}.$$

Then from equation (2), $b^2 = 1/2$, so $b = \pm 1/\sqrt{2}$. Thus the equations of the two required tangent lines are

$$y = \frac{2-2a}{b}(x-a) + b,$$

that is

$$y = \sqrt{2}\left(x - \frac{1}{2}\right) + \frac{1}{\sqrt{2}} \text{ or } y = -\sqrt{2}\left(x - \frac{1}{2}\right) - \frac{1}{\sqrt{2}};$$

or simply

$$y = \pm\sqrt{2}\,x.$$

3. [8 marks] Given that $\theta = \cos^{-1}\left(-\frac{2}{5}\right)$ find the *exact* values of the following:

(a) [2 marks] $\sin\theta$

Solution: we have $\cos\theta = -2/5$ and $\pi/2 < \theta < \pi$, so $\sin\theta > 0$. Consequently

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{-2}{5}\right)^2} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}.$$

(b) [3 marks] $\sin(2\theta)$

Solution: use the appropriate double angle formula.

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right) = -\frac{4\sqrt{21}}{25}.$$

(c) [3 marks] $\cos\left(\theta - \frac{\pi}{3}\right)$

Solution: use the formula for $\cos(\alpha - \beta)$.

$$\begin{aligned}\cos\left(\theta - \frac{\pi}{3}\right) &= \cos\theta\cos\left(\frac{\pi}{3}\right) + \sin\theta\sin\left(\frac{\pi}{3}\right)\\ &= -\frac{2}{5}\cdot\frac{1}{2} + \frac{\sqrt{21}}{5}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{3\sqrt{7}-2}{10}\end{aligned}$$

4. [8 marks; 4 marks each] Find the following limits:

(a) $\lim_{x\to -1} \frac{x^3+5x^2-x-5}{x^2-3x-4}$

Solution: factor the numerator and the denominator.

$$\begin{aligned}
\lim_{x\to -1} \frac{x^3+5x^2-x-5}{x^2-3x-4} &= \lim_{x\to -1} \frac{(x+1)(x^2+4x-5)}{(x+1)(x-4)} \\
&= \lim_{x\to -1} \frac{x^2+4x-5}{x-4} \\
&= \frac{1-4-5}{-1-4} \\
&= \frac{8}{5}
\end{aligned}$$

(b) $\lim_{x\to 0} \frac{4-\sqrt{3x+16}}{2x}$

Solution: rationalize the numerator.

$$\begin{aligned}
\lim_{x\to 0} \frac{4-\sqrt{3x+16}}{2x} &= \lim_{x\to 0} \frac{(4-\sqrt{3x+16})(4+\sqrt{3x+16})}{2x(4+\sqrt{3x+16})} \\
&= \lim_{x\to 0} \frac{16-(3x+16)}{2x\,(4+\sqrt{3x+16})} \\
&= -\frac{3}{2} \lim_{x\to 0} \frac{x}{x\,(4+\sqrt{3x+16})} \\
&= -\frac{3}{2} \lim_{x\to 0} \frac{1}{4+\sqrt{3x+16}} \\
&= -\frac{3}{2} \frac{1}{(4+4)} \\
&= -\frac{3}{16}
\end{aligned}$$

5. [8 marks; 4 marks each] Find the following limits:

(a) $\displaystyle\lim_{x\to 0}\frac{\tan(5x)}{\sin(2x)}$

Solution: reduce the limit to applications of the basic trig limit $\displaystyle\lim_{h\to 0}\frac{\sin h}{h}=1$.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(5x)}{\sin(2x)} &= \frac{5}{2}\lim_{x\to 0}\frac{1}{\cos(5x)}\lim_{x\to 0}\frac{\sin(5x)}{5x}\lim_{x\to 0}\frac{2x}{\sin(2x)}\\ &= \frac{5}{2}\cdot\frac{1}{1}\cdot 1\cdot\frac{1}{1}\\ &= \frac{5}{2}\end{aligned}$$

(b) $\displaystyle\lim_{x\to 0^-}\tan^{-1}\left(\frac{1+x}{3x}\right)$.

Solution: let $u=\dfrac{1+x}{3x}$; as $x\to 0^-$, $u\to-\infty$. So

$$\begin{aligned}\lim_{x\to 0^-}\tan^{-1}\left(\frac{1+x}{3x}\right) &= \lim_{u\to-\infty}\tan^{-1}u\\ &= -\frac{\pi}{2}\end{aligned}$$

6. [8 marks; 4 marks for each part] Suppose $f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 1 \\ x & \text{if} \quad -1 < x < 1 \\ \dfrac{3x}{\sqrt{x^2+8}} & \text{if} \quad x \leq -1 \end{cases}$

(a) Show that $f(x)$ is continuous at both $x = 1$ and $x = -1$.

Solution: observe that $f(1) = 1$ and $f(-1) = -1$. At $x = 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1 \text{ and } \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1.$$

So

$$\lim_{x\to 1} f(x) = f(1)$$

and f is continuous at $x = 1$. At $x = -1$:

$$\lim_{x\to -1^-} f(x) = \lim_{x\to 1^+} \frac{3x}{\sqrt{x^2+8}} = \frac{-3}{\sqrt{9}} = -1 \text{ and } \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1.$$

So

$$\lim_{x\to -1} f(x) = f(-1)$$

and f is continuous at $x = -1$.

(b) Find the formula for $f^{-1}(x)$. (Assume that f is invertible.)

Solution: for $x \geq 1$, take $f^{-1}(x) = \sqrt{x}$; for $-1 < x < 1$, take $f^{-1}(x) = x$.

For $x \leq -1$ and $y < 0$:

$$x = \frac{3y}{\sqrt{y^2+8}}$$
$$\Rightarrow \; x^2(y^2+8) = 9y^2$$
$$\Rightarrow \; 8x^2 = (9-x^2)y^2$$
$$\Rightarrow \; y = \frac{\sqrt{8}x}{\sqrt{9-x^2}}$$

for $x > -3$.

So $f^{-1}(x) = \begin{cases} \sqrt{x} & \text{if} \quad x \geq 1 \\ x & \text{if} \quad -1 < x < 1 \\ \dfrac{\sqrt{8}x}{\sqrt{9-x^2}} & \text{if} \quad -3 < x \leq -1 \end{cases}$

7.(a) [3 marks] Use the Intermediate Value Theorem to explain, clearly and concisely, why the equation $x^3 - x - 1 = 0$ has at least one solution in the interval $[1, 2]$.

Solution: let $f(x) = x^3 - x - 1$, which is a polynomial function and so is continuous for all x. Now consider $f(x)$ on the closed interval $[1, 2]$:

$$f(1) = -1 < 0 \text{ and } f(2) = 5 > 0.$$

So by the Intermediate Value Theorem there is a number $c \in (1, 2)$ such that

$$f(c) = 0 \Leftrightarrow c^3 - c - 1 = 0.$$

7.(b) [3 marks] Suppose $f(x)$ is a function such that $1 - x^2 \leq f(x) \leq \cos x$ for all x in the interval $(-\pi/2, \pi/2)$. Find $\lim_{x \to 0} f(x)$.

Solution: use the Squeezing Theorem. Since

$$\lim_{x \to 0}(1 - x^2) = 1 - 0 = 1 \text{ and } \lim_{x \to 0} \cos x = \cos 0 = 1$$

and

$$1 - x^2 \leq f(x) \leq \cos x,$$

for all $x \in (-\pi/2, \pi/2)$ it follows from the Squeezing Theorem that

$$\lim_{x \to 0} f(x) = 1.$$

8. [8 marks] Find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(x, y) = (1, 1)$ if $\quad x^2 - 4y^2 = 7xy - 10.$

Solution: differentiate implicitly.

$$2x - 8y\frac{dy}{dx} = 7y + 7x\frac{dy}{dx}$$

At $(x, y) = (1, 1)$, this becomes

$$2 - 8\frac{dy}{dx} = 7 + 7\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}.$$

To find $\frac{d^2y}{dx^2}$, differentiate implicitly once more:

$$2 - 8\frac{dy}{dx}\frac{dy}{dx} - 8y\frac{d^2y}{dx^2} = 7\frac{dy}{dx} + 7\frac{dy}{dx} + 7x\frac{d^2y}{dx^2}.$$

At $(x, y) = (1, 1)$, $\frac{dy}{dx} = -\frac{1}{3}$, so

$$2 - 8\left(-\frac{1}{3}\right)^2 - 8(1)\frac{d^2y}{dx^2} = 14\left(-\frac{1}{3}\right) + 7\frac{d^2y}{dx^2} \Leftrightarrow \frac{52}{9} = 15\frac{d^2y}{dx^2} \Leftrightarrow \frac{d^2y}{dx^2} = \frac{52}{135}$$