University of Toronto

SOLUTIONS to MAT 186H1F TERM TEST 1

of **Thursday, October 4, 2007**

Duration: 60 minutes

TOTAL MARKS: 50

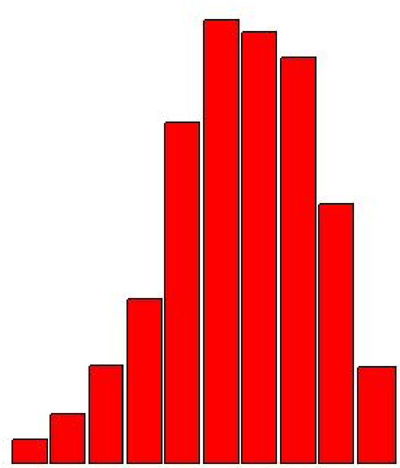
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

- Questions 1, 2, 4, 6 and 7 are considered completely routine, straightforward calculations. This is the 66% of the test that should have allowed everybody to pass the test.
- Questions 3 and 5(c) are checking if you know what the Intermediate Value Property and the Squeeze Law say, and if you can use them. The actual calculations involved are very simple.
- Questions 5(a) and 5(b) require a little manipulation to reduce both questions to the basic trigonometric limit
$$\lim_{h\to 0}\frac{\sin h}{h}=1.$$
- Question 8 requires some thought to set up, but it requires no more calculus than what you learned in high school.
- Some alternate solutions are included at the end. However, L'Hopital's Rule was not permitted on this test.

Breakdown of Results: 566 students wrote this test. The marks ranged from 0% to 100%, and the average was 58.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.6%
A	15.7%	80-89%	11.1%
B	17.5%	70-79%	17.5%
C	18.6%	60-69%	18.6%
D	19.1%	50-59%	19.1%
F	29.2%	40-49%	14.7%
		30-39%	7.1%
		20-29%	4.2%
		10-19%	2.1%
		0-9%	1.1 %



1. [8 marks] Suppose $\cos x = -\dfrac{1}{3}$ and $\sin x > 0$. Find the exact values of the following:

(a) [2 marks] $\sin x$

Solution:

$$\begin{aligned} \sin x &= \sqrt{1-\cos^2 x} \\ &= \sqrt{1-\left(-\frac{1}{3}\right)^2} \\ &= \frac{2\sqrt{2}}{3} \end{aligned}$$

(b) [3 marks] $\cos\left(x+\dfrac{\pi}{6}\right)$

Solution:

$$\begin{aligned} \cos\left(x+\frac{\pi}{6}\right) &= \cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6} \\ &= -\frac{1}{3}\frac{\sqrt{3}}{2} - \frac{2\sqrt{2}}{3}\frac{1}{2} \\ &= -\frac{\sqrt{3}+2\sqrt{2}}{6} \end{aligned}$$

(c) [3 marks] $\cos(2x)$

Solution:

$$\begin{aligned} \cos(2x) &= 2\cos^2 x - 1 \\ &= 2\left(-\frac{1}{3}\right)^2 - 1 \\ &= -\frac{7}{9} \end{aligned}$$

2. [4 marks] Find all the solutions x in the interval $[0, \pi]$ to the equation

$$4 \sin^2 x \cos^2 x = 1.$$

Solution:

$$\begin{aligned} 4 \sin^2 x \cos^2 x = 1 &\Leftrightarrow (2 \sin x \cos x)^2 = 1 \\ &\Leftrightarrow (\sin(2x))^2 = 1 \\ &\Leftrightarrow \sin(2x) = 1 \text{ or } \sin(2x) = -1 \end{aligned}$$

If $x \in [0, \pi]$, then $2x \in [0, 2\pi]$. Take

$$2x = \frac{\pi}{2} \text{ or } 2x = \frac{3\pi}{2} \Leftrightarrow x = \frac{\pi}{4} \text{ or } x = \frac{3\pi}{4}.$$

3. [3 marks] Use the Intermediate Value Property to explain why the equation $\cos x = x$ has a solution in the interval $[0, \pi/2]$.

Solution: Let $f(x) = \cos x - x$, which is a continuous function on $[0, \pi/2]$. Observe that

$$f(0) = 1 - 0 = 1 > 0 \text{ and } f\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2} = -\frac{\pi}{2} < 0.$$

So by the Intermediate Value Property, there is a $c \in (0, \pi/2)$ such that

$$f(c) = 0 \Leftrightarrow \cos c - c = 0 \Leftrightarrow \cos c = c.$$

4. [8 marks] Find the following limits.

(a) [4 marks] $\lim\limits_{x \to -2} \dfrac{x^3+8}{x+2}$

Solution:

$$\begin{aligned}
\lim_{x \to -2} \frac{x^3+8}{x+2} &= \lim_{x \to -2} \frac{(x+2)(x^2-2x+4)}{x+2} \\
&= \lim_{x \to -2} \left(x^2-2x+4\right) \\
&= (-2)^2 - 2(-2) + 4 \\
&= 12
\end{aligned}$$

(b) [4 marks] $\lim\limits_{x \to 0} \dfrac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

Solution:

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x+1}-1}{\sqrt{x+4}-2} &= \lim_{x \to 0} \left(\frac{\sqrt{x+1}-1}{\sqrt{x+4}-2}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right) \\
&= \lim_{x \to 0} \left(\frac{x}{\sqrt{x+1}+1}\right)\left(\frac{\sqrt{x+4}+2}{x}\right) \\
&= \lim_{x \to 0} \frac{\sqrt{x+4}+2}{\sqrt{x+1}+1} \\
&= \frac{2+2}{1+1} \\
&= 2
\end{aligned}$$

5. [8 marks] Find the following limits.

(a) [2 marks] $\lim_{h\to 0} \dfrac{\sin(3h)}{h}$

Solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{\sin(3h)}{h} &= \lim_{h\to 0} \frac{3\sin(3h)}{3h} \\
&= \lim_{k\to 0} \frac{3\sin k}{k}, \text{ with } k = 3h \\
&= 3\lim_{k\to 0} \frac{\sin k}{k} \\
&= 3\cdot 1 \\
&= 3
\end{aligned}$$

(b) [3 marks] $\lim_{x\to \pi} \dfrac{x-\pi}{\sin x}$

Solution:

$$\begin{aligned}
\lim_{x\to \pi} \frac{x-\pi}{\sin x} &= \lim_{h\to 0} \frac{h}{\sin(h+\pi)}, \text{ with } h = x-\pi \\
&= \lim_{h\to 0} \frac{h}{-\sin h} \\
&= -\lim_{h\to 0} \frac{h}{\sin h} \\
&= (-1)\cdot 1 \\
&= -1
\end{aligned}$$

(c) [3 marks] $\lim_{x\to 0} x^2 \cos\left(\dfrac{1}{x}\right)$

Solution: Use Squeeze Law. For $x \neq 0$,

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2.$$

Since both

$$\lim_{x\to 0}(-x^2) = 0 \text{ and } \lim_{x\to 0} x^2 = 0,$$

if follows that

$$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0$$

as well.

6. [7 marks] Let

$$f(x) = \frac{x^2 - 16}{x^2 - 3x - 4}.$$

Find all points where the function is not defined, and at each such point $x = a$, calculate both

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^+} f(x),$$

or explain why they do not exist.

Solution: $f(x)$ is undefined if and only if

$$x^2 - 3x - 4 = 0 \Leftrightarrow (x-4)(x+1) = 0 \Leftrightarrow x = 4 \text{ or } x = -1.$$

At $x = 4$,

$$\begin{aligned} \lim_{x \to 4} f(x) &= \lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)(x+1)} \\ &= \lim_{x \to 4} \frac{x+4}{x+1} \\ &= \frac{8}{5}; \end{aligned}$$

so both

$$\lim_{x \to 4^-} f(x) = \frac{8}{5} \text{ and } \lim_{x \to 4^+} f(x) = \frac{8}{5}.$$

At $x = -1$,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x+4}{x+1} = -\infty, \text{ since } x + 1 \to 0^-$$

and

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x+4}{x+1} = +\infty, \text{ since } x + 1 \to 0^+.$$

7. [6 marks] Find the equation of the tangent line to the graph of

$$y = f(x) = \frac{3x^3 + 1}{x^2 + x},$$

at the point $(x, y) = (1, 2)$. Put your answer in the form $y = mx + b$.

Solution: Use Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{9x^2(x^2 + x) - (2x + 1)(3x^3 + 1)}{(x^2 + x)^2} \\ \Rightarrow f'(1) &= \frac{9(2) - 3(4)}{2^2} = \frac{3}{2}. \end{aligned}$$

The equation of the tangent line to $y = f(x)$ at $x = 1$ is

$$\begin{aligned} & \frac{y-2}{x-1} = f'(1) = \frac{3}{2} \\ \Leftrightarrow \quad & y - 2 = \frac{3}{2}(x - 1) \\ \Leftrightarrow \quad & y = \frac{3}{2}x - \frac{3}{2} + 2 \\ \Leftrightarrow \quad & y = \frac{3}{2}x + \frac{1}{2} \end{aligned}$$

8. [6 marks] Find the equation of the line passing through the origin and normal to the graph of the function

$$y = f(x) = x + \frac{1}{x}.$$

Put your answer in the form $y = mx + b$.

Solution:

$$f'(x) = 1 - \frac{1}{x^2}.$$

Let the point of intersection of the line and the graph be $(a, f(a))$; then the equation of the normal line to the graph of $y = f(x)$ at $x = a$ is

$$\frac{y - f(a)}{x - a} = -\frac{1}{f'(a)} = -\left(1 - \frac{1}{a^2}\right)^{-1} = -\frac{a^2}{a^2 - 1}$$

Since the normal line passes through $(x, y) = (0, 0)$, substitute $x = 0, y = 0$ into the previous equation:

$$\begin{aligned}
\frac{0 - f(a)}{0 - a} = -\frac{a^2}{a^2 - 1} \quad &\Leftrightarrow \quad \frac{f(a)}{a} = -\frac{a^2}{a^2 - 1} \\
&\Leftrightarrow \quad (a^2 - 1)\left(a + \frac{1}{a}\right) = -a^3 \\
&\Leftrightarrow \quad a^3 - a + a - \frac{1}{a} = -a^3 \\
&\Leftrightarrow \quad 2a^3 = \frac{1}{a} \\
&\Leftrightarrow \quad a^4 = \frac{1}{2} \\
&\Rightarrow \quad a^2 = \frac{1}{\sqrt{2}}, \text{ since } a^2 \geq 0.
\end{aligned}$$

Thus

$$f'(a) = 1 - \frac{1}{a^2} = 1 - \sqrt{2}$$

and for the normal line

$$m = -\frac{1}{f'(a)} = -\frac{1}{1 - \sqrt{2}} = 1 + \sqrt{2}.$$

So the equation of the line is

$$y = (1 + \sqrt{2})\, x,$$

since $b = 0$ if a line passes through the origin.

Some alternate solutions:

1.(c) You could use either of the alternatives

$$\cos(2x) = 1 - 2\sin^2 x \text{ or } \cos(2x) = \cos^2 x - \sin^2 x.$$

2. Substitute $\cos^2 x = 1 - \sin^2 x$ and the equation becomes

$$\begin{aligned}
4\sin^2 x\cos^2 x = 1 \Leftrightarrow \quad & 4\sin^2 x\,(1-\sin^2 x) = 1 \\
\Leftrightarrow \quad & 4\sin^2 x - 4\sin^4 x = 1 \\
\Leftrightarrow \quad & 4\sin^4 x - 4\sin^2 x + 1 = 0 \\
\Leftrightarrow \quad & (2\sin^2 x - 1)^2 = 0 \\
\Leftrightarrow \quad & \sin^2 x = \frac{1}{2} \\
\Leftrightarrow \quad & \sin x = \pm\frac{1}{\sqrt{2}} \\
\Rightarrow \quad & x = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}
\end{aligned}$$

8. Let the line be $y = mx$. Find the intersection of the line and the function $y = f(x)$:

$$\begin{cases} y &= x + \dfrac{1}{x} \\ y &= m\,x \end{cases}$$

At the intersection point

$$mx^2 = x^2 + 1 \Leftrightarrow x^2 = \frac{1}{m-1};$$

$$\text{so } m > 1, \text{ and } f'(x) = 1 - \frac{1}{x^2} = 1 - (m-1) = 2 - m.$$

For the line to be normal to the graph at the point of intersection, we must have

$$\begin{aligned}
2 - m = -\frac{1}{m} \quad \Leftrightarrow \quad & m^2 - 2m - 1 = 0 \\
\Leftrightarrow \quad & m = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} \\
\Rightarrow \quad & m = 1 + \sqrt{2}, \text{ since } m > 1.
\end{aligned}$$

Thus the line is

$$y = (1 + \sqrt{2})\,x.$$