MAT186H1F - Calculus I - Fall 2018

Solutions to Term Test 2 - November 20, 2018

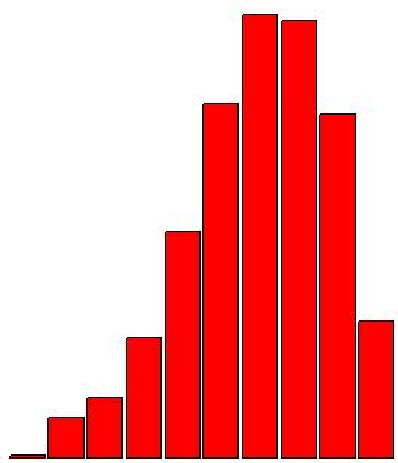
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- The range on every question was 0 to 10; only Question 4 had a failing average. There were no perfect papers, but three students received 79/80.
- In Question 5 a common misconception was that "$f''(x) = 0 \Rightarrow f$ has an inflection point at x." This is false: consider $f(x) = x^4$. $f''(0) = 0$, but the graph of $y = x^4$ has no inflection points.
- Many students made Question 6 far more complicated than it really is.
- Question 7(b) was part of a tutorial question. Still, some students didn't know how to approach it.
- Only 23 students got more than 6 on Question 4; only one had 10/10.

Breakdown of Results: 757 registered students wrote this test. The marks ranged from 8.75% to 98.75%, and the average was 64.1%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.3%
A	22.2%	80-89%	15.9%
B	20.2%	70-79%	20.2%
C	20.5%	60-69%	20.5%
D	16.4%	50-59%	16.4%
F	20.6%	40-49%	10.4%
		30-39%	5.5%
		20-29%	2.8%
		10-19%	1.8%
		0-9%	0.1%



1. [avg: 7.96/10] Find the following limits:

   (a) [4 marks] $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{\tan^{-1}(2x)}$

   Solution: the limit is in the 0/0 form. Use L'Hopital's Rule.

$$\lim_{x\to 0}\frac{\sin(5x)}{\tan^{-1}(2x)}=\lim_{x\to 0}\frac{5\cos(5x)}{\dfrac{2}{1+4x^2}}=\frac{5}{2}$$

   (b) [6 marks] $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right)$

   Solution: the limit is in the $\infty-\infty$ form. Get a common denominator to put it into the 0/0 form, and then use L'Hopital's Rule.

$$\begin{aligned}
\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right) &= \lim_{x\to 0^+}\frac{\ln(1+x)-x}{x\ln(1+x)},\ \text{in 0/0 form}\\
\text{(apply L'H)} &= \lim_{x\to 0^+}\frac{\dfrac{1}{1+x}-1}{\ln(1+x)+\dfrac{x}{1+x}},\ \text{still in 0/0 form}\\
&= \lim_{x\to 0^+}\frac{\dfrac{1}{1+x}-1}{\ln(1+x)+1-\dfrac{1}{1+x}}\\
\text{(apply L'H)} &= \lim_{x\to 0^+}\frac{\dfrac{-1}{(1+x)^2}}{\dfrac{1}{1+x}+\dfrac{1}{(1+x)^2}}\\
\text{(substitute } x=0) &= -\frac{1}{2}
\end{aligned}$$

2. [avg: 7.99/10] Find the following derivatives:

(a) [4 marks] $F'(-2)$ if $F(x) = \displaystyle\int_2^{x^3} \sqrt{t^{4/3}+9}\,dt$

Solution: use the Fundamental Theorem, and the chain rule.

$$F'(x) = \sqrt{(x^3)^{4/3}+9}\,(3x^2) = 3x^2\sqrt{x^4+9}.$$

So

$$F'(-2) = (3(-2)^2)\sqrt{(-2)^4+9} = 12\sqrt{16+9} = 12\times 5 = 60.$$

(b) [6 marks] both $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(x,y)=(1,0)$ if $e^y = x^2 - xy$.

Solution: differentiate implicitly:

$$e^y\frac{dy}{dx} = 2x - y - x\frac{dy}{dx} \quad (*).$$

So at $(x,y)=(1,0)$ we get

$$1\frac{dy}{dx} = 2 - 0 - \frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = 1.$$

Now differentiate $(*)$ implicitly again:

$$e^y\left(\frac{dy}{dx}\right)^2 + e^y\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} - \frac{dy}{dx} - x\frac{d^2y}{dx^2}.$$

At $(x,y)=(1,0)$, we know $\dfrac{dy}{dx}=1$, so we have

$$1(1)^2 + 1\frac{d^2y}{dx^2} = 2 - 1 - 1 - \frac{d^2y}{dx^2} \Leftrightarrow \frac{d^2y}{dx^2} = -\frac{1}{2}.$$

3. [avg: 6.99/10] Compute the following integrals:

(a) [4 marks] $\displaystyle\int_0^2 x\, e^{(x^2)}\, dx$

Solution: let $u = x^2$, so $du = 2x\, dx$. Then

$$\int_0^2 x\, e^{(x^2)}\, dx = \frac{1}{2}\int_0^4 e^u\, du = \frac{1}{2}\left[e^u\right]_0^4 = \frac{e^4-1}{2}.$$

(b) [6 marks] $\displaystyle\int_0^2 x^2\, e^x\, dx.$

Solution: integrate by parts twice. Start with $u = x^2$, $dv = e^x\, dx$. Then $du = 2x\, dx$ and $v = e^x$. So

$$\begin{aligned}
\int_0^2 x^2\, e^x\, dx &= \left[x^2\, e^x\right]_0^2 - \int_0^2 (e^x)\,(2x)\, dx \\
&= 4e^2 - 2\int_0^2 x\, e^x\, dx \\
\text{(use parts again with } s = x, dt = e^x\, dx) \quad &= 4e^2 - 2[xe^x]_0^2 + 2\int_0^2 e^x\, dx \\
&= 4e^2 - 4e^2 + 2[e^x]_0^2 \\
&= 2(e^2-1)
\end{aligned}$$

4. [avg: 3.57/10]

4.(a) Suppose that $f''(x) = 0$ for all $x \in \mathbb{R}$.

(i) [2 marks] Find the general form of $f(x)$.

Solution: $f'(x) = C$; $f(x) = \displaystyle\int C\,dx = Cx + D$, for constants C and D.

(ii) [4 marks] Show that the average value of f on the interval $[a, b]$ is equal to $f\left(\dfrac{a+b}{2}\right)$.

Solution: on the one hand, $f\left(\dfrac{a+b}{2}\right) = \dfrac{C}{2}(a+b) + D$; on the other hand, the average value of f is the same:

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b (Cx+D)\,dx &= \frac{1}{b-a}\left[\frac{Cx^2}{2} + Dx\right]_a^b \\
&= \frac{1}{b-a}\left(\frac{Cb^2}{2} + Db - \frac{Ca^2}{2} - Da\right) \\
&= \frac{1}{b-a}\left(\frac{C(b^2-a^2)}{2} + D(b-a)\right) \\
&= \frac{C}{2}(b+a) + D
\end{aligned}$$

4.(b) [4 marks] Suppose that g has a continuous second derivative and that for any interval $[a, b]$ the average value of g on $[a, b]$ is equal to $g\left(\dfrac{a+b}{2}\right)$. Prove that $g''(x) = 0$ for all $x \in \mathbb{R}$. (Don't waste time on this question; its short but tricky. Save it for after you've finished the rest of the test.)

Solution: consider the interval $[-a, x]$. It is given that

$$\frac{1}{x+a}\int_{-a}^{x} g(t)\,dt = g\left(\frac{x-a}{2}\right) \Leftrightarrow \int_{-a}^{x} g(t)\,dt = (x+a)\,g\left(\frac{x-a}{2}\right).$$

Differentiate with respect to x :

$$g(x) = 1 \cdot g\left(\frac{x-a}{2}\right) + (x+a)\,g'\left(\frac{x-a}{2}\right)\frac{1}{2}$$

Let $x = a$ to obtain

$$g(a) = g(0) + (2a)\frac{g'(0)}{2} = g(0) + ag'(0).$$

Since this is true for all a, g is a linear function, and so its second derivative is zero.

Alternate Solutions: see page 10

5. [avg: 6.76/10] Let $f(x) = x\,e^{-x/2}$.

(a) [3 marks] Find all the critical points of f and at each critical point determine if f has a maximum or minimum value at the point.

Solution:

$$f'(x) = e^{-x/2} - \frac{x}{2}e^{-x/2} = e^{-x/2}\left(1 - \frac{x}{2}\right)$$

The only critical point is $x = 2$. Since

$$f'(x) > 0 \Leftrightarrow x < 2 \text{ and } f'(x) < 0 \Leftrightarrow x > 2,$$

f has an (absolute) maximum at $(x, y) = (2, 2/e)$.

(b) [3 marks] Find all the inflection points of f, if any.

Solution:

$$f''(x) = -\frac{1}{2}e^{-x/2} - \frac{1}{2}e^{-x/2} + \frac{x}{4}e^{-x/2} = e^{-x/2}\left(\frac{x}{4} - 1\right).$$

Since

$$f''(x) > 0 \Leftrightarrow x > 4 \text{ and } f''(x) < 0 \Leftrightarrow x < 4$$

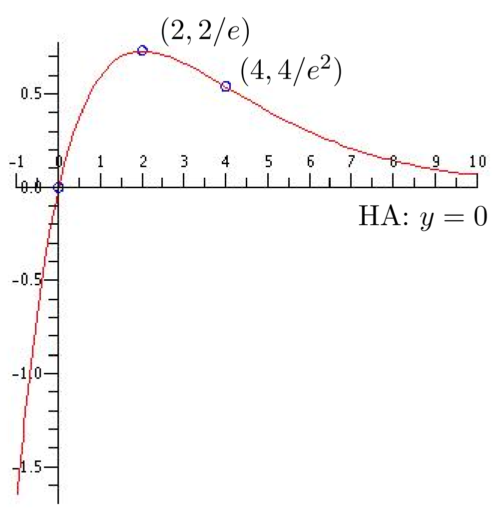
f has exactly one inflection point, at $(x, y) = (4, 4/e^2)$.

(c) [4 marks] Sketch the graph of f, labeling all critical points, inflection points, and asymptotes, if any.

Solution: there is a horizontal asymptote at $y = 0$ since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^{x/2}} \overbrace{=}^{L'H} \lim_{x\to\infty} \frac{2}{e^{x/2}} = 0.$$

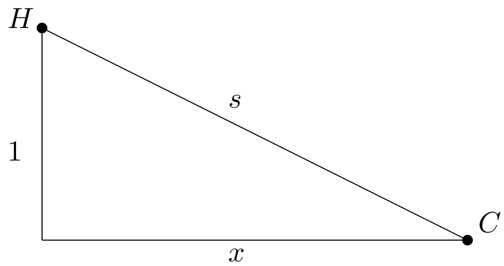
The graph is



To get full marks you must label the maximum point, the inflection point, and the horizontal asymptote; your graph must go through the point $(0, 0)$; and it must be increasing/decreasing on the correct intervals, and concave up/down on the proper intervals.

6. [avg: 5.11/10] A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car.

Solution: let x be the distance (in km) between the car, C, on the road and the point directly below the helicopter, H. Let s be the distance (in km) between the helicopter and the car. (See diagram.)



We have

$$x^2 + 1 = s^2 \quad \Rightarrow \quad 2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$
$$\Rightarrow \quad \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}.$$

At the instant when $s = 2$ and $\frac{ds}{dt} = -250$ we have $x = \sqrt{s^2 - 1} = \sqrt{3}$. So at this moment,

$$\frac{dx}{dt} = \frac{2}{\sqrt{3}}(-250) = -\frac{500}{\sqrt{3}},$$

where the negative sign indicates the distance between the car and the point on the road directly below the helicopter is decreasing.

But the point beneath the helicopter is moving towards the car at a speed of 200 km/hr, so the speed of the car at the moment in question is

$$\frac{500}{\sqrt{3}} - 200$$

kilometres per hour. (For interest, this is approximately 88.7 km/hr.)

7. [avg: 6.96/10]

7.(a) [4 marks] Use a linear approximation to approximate $\ln(1.03)$.

Solution: take $a = 1$. Then the tangent line approximation to $\ln x$ at $x = a$ is:

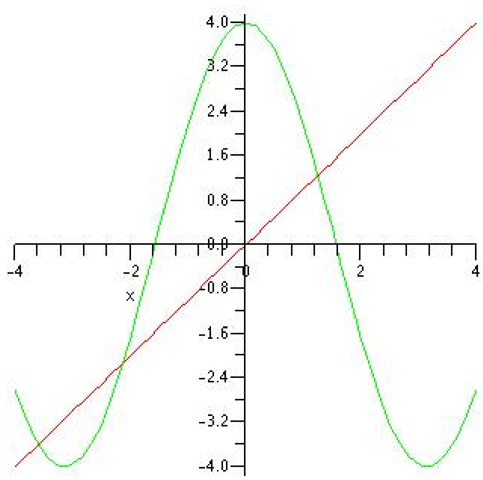
$$\ln x \approx \ln 1 + \ln'(1)\,(x-1) = 0 + \frac{1}{1}(x-1) = x - 1,$$

so

$$\ln 1.03 \approx 0.03.$$

7.(b) [6 marks] Use Newton's method to approximate the positive solution to the equation $x = 4\cos x$ correct to 5 decimal places. (Note: your calculator must be in radian mode!)

Solution: the graphs of $y = x$ and $y = 4\cos x$ are given below. Let $f(x) = x - 4\cos x$. Then



$$f'(x) = 1 + 4\sin x$$

and Newton's recursive formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n - 4\cos x_n}{1 + 4\sin x_n} \\ &= \frac{4x_n \sin x_n + 4\cos x_n}{1 + 4\sin x_n} \end{aligned}$$

for $n \geq 0$. Your first choice, x_0, should be something in the interval $[1, 2]$ to (hopefully) approximate the positive root to $f(x) = 0$. Take $x_0 = 1.5$.

Then use your calculator to find

$$x_1 = 1.256100985\ldots$$
$$x_2 = 1.252355051\ldots$$
$$x_3 = 1.252353234\ldots$$
$$x_4 = 1.252353234\ldots$$

So correct to 5 decimal places, the positive solution to $x = 4\cos x$ is 1.25235

Note: instead of using a graphical approach you could observe that $f(1) = 1 - 4\cos 1 \approx -1.16 < 0$ and that $f(2) = 2 - 4\cos 2 \approx 3.66 > 0$. Thus by good old IVT, there is a solution to the equation $f(x) = 0$ in the interval $[1, 2]$.

8. [avg: 5.93/10] A company is designing propane tanks that are cylindrical with hemispherical ends. Each of the tanks is to hold 1000 cubic meters of gas. The ends are more expensive to make, costing \$8 per square meter, while the cylindrical barrel between the ends costs only \$3 per square meter to make. Determine the dimensions of the tank that minimize the cost to make it. What is the minimum cost? (The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and its surface area is $S = 4\pi r^2$.)

Solution: let r be the radius (in m) of the hemispherical ends; let h be the height (in m) of the cylindrical barrel. Then the volume of the tank is

$$V = \pi r^2 h + \frac{4}{3}\pi r^3.$$

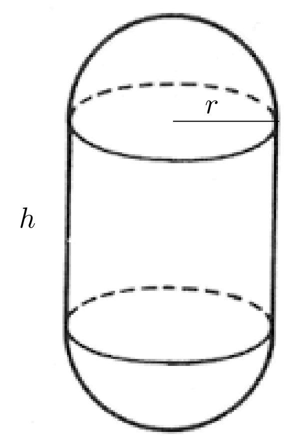
Since $V = 1000$ we can solve for h in terms of r :

$$h = \frac{1000 - \frac{4}{3}\pi r^3}{\pi r^2} = \frac{1000}{\pi r^2} - \frac{4r}{3}.$$

Let C be the cost of making the tank. We have

$$C = 3(2\pi r h) + 8(4\pi r^2),$$

since surface area of the cylindrical barrel is $2\pi r h$ and the surface area of a sphere is $4\pi r^2$. Substituting for h in terms of r we get



$$C = 6\pi r\left(\frac{1000}{\pi r^2} - \frac{4r}{3}\right) + 32\pi r^2 = \frac{6000}{r} + 24\pi r^2.$$

The problem is to minimize C for $r > 0$. The first and second derivatives of C are

$$\frac{dC}{dr} = -\frac{6000}{r^2} + 48\pi r \text{ and } \frac{d^2C}{dr^2} = \frac{12000}{r^3} + 48\pi.$$

Since the second derivative is always positive, the critical point of C will give a minimum value of C. We have

$$\frac{dC}{dr} = 0 \Rightarrow \pi r^3 = \frac{6000}{48} = 125 \Rightarrow r = \frac{5}{\pi^{1/3}}.$$

Then

$$h = \frac{1000\pi^{2/3}}{25\pi} - \frac{20}{3\pi^{1/3}} = \frac{100}{3\,\pi^{1/3}},$$

and the minimum cost of construction is

$$C = 1000\,\pi^{1/3} + 800\,\pi^{1/3} = 1800\,\pi^{1/3}.$$

(Aside: $C \approx \$2,636.26$)

Alternate Solutions:

4(b): similar to first solution, just a different interval. Consider the interval $[a, x]$. It is given that

$$\frac{1}{x-a}\int_a^x g(t)\,dt = g\left(\frac{x+a}{2}\right) \Leftrightarrow \int_a^x g(t)\,dt = (x-a)\,g\left(\frac{x+a}{2}\right).$$

Differentiate with respect to x :

$$g(x) = 1 \cdot g\left(\frac{x-a}{2}\right) + (x-a)\,g'\left(\frac{x+a}{2}\right)\frac{1}{2}$$

Let $x = -a$ to obtain

$$g(-a) = g(0) + (-2a)\frac{g'(0)}{2} = g(0) - a\,g'(0).$$

Since this is true for all $b = -a$, g is a linear function, namely $g(b) = g(0) + bg'(0)$, and so its second derivative is zero.

4(b): this is a more intuitive solution. Use concavity. Suppose there is a z such that $g''(z) > 0$. By continuity, there is an interval $[a, b]$ containing z such that $g''(x) > 0$ for all $x \in [a, b]$. Consider the tangent line to g at the point $x = (a + b)/2$:

$$L(x) = g((a+b)/2) + g'((a+b)/2)(x - (a+b)/2).$$

Since L is linear, $L''(x) = 0$ and 4.(a) implies that

$$g((a+b)/2) = L((a+b)/2) = \frac{1}{b-a}\int_a^b L(x)\,dx. \quad (1)$$

But, since g is concave up on the interval $[a, b]$, the graph of g is above the graph of L, so

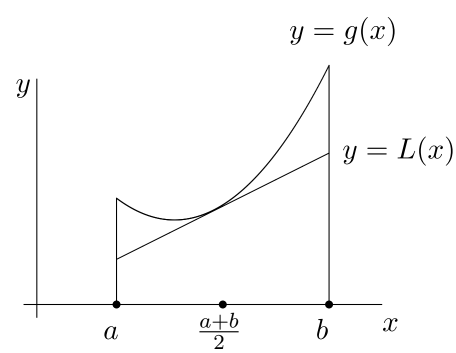
$$\int_a^b L(x)\,dx < \int_a^b g(x)\,dx. \quad (2)$$

By hypothesis,

$$\frac{1}{b-a}\int_a^b g(x)\,dx = g((a+b)/2). \quad (3)$$

Comparing (1), (2) and (3) we conclude that

$$g((a+b)/2) < g((a+b)/2),$$



which is a contradiction, since no number can be less than itself. Similarly, if $g''(z) < 0$, for some value of z, you can also derive a contradiction. So the only possibility left is that $g''(z) = 0$, for all z.

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