MAT186H1F - Calculus I - Fall 2017

Solutions to Term Test 2 - November 21, 2017

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

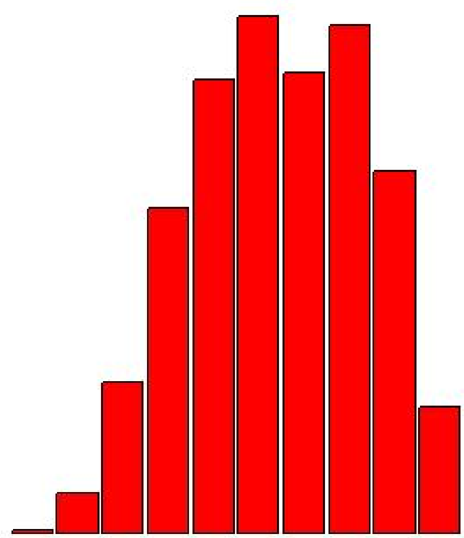
Comments:

1. Every question had a passing average, except for Question 8. But to pass Question 8 all you needed to do was draw a diagram, define the function to be minimized, and calculate its derivative. Everyone should have been able to do that. The rest of the question *is* more difficult, especially confirming that your crucial point gives the minimum value.

2. One third of the class failed this test! That is hard to believe, since so many questions came right out of WeBWorK and/or the text book. In any event, based on each term test counting 20%, and assuming an average combined mark of 9 out of 10 for MDT and WeBWorK, the average term mark in the course is (presently) about 35.4 out of 50, or approximately 70.8 %.

Breakdown of Results: 742 registered students wrote this test. The marks ranged from 2.5% to 98.75%, and the average was (only) 58,9%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.3%
A	16.6%	80-89%	12.3%
B	17.3%	70-79%	17.3%
C	15.6%	60-69%	15.6%
D	17.5%	50-59%	17.5%
F	33.0%	40-49%	15.4%
		30-39%	11.1%
		20-29%	5.1%
		10-19%	1.3%
		0-9%	0.1%



1. [avg: 7.05/10] Find the following limits:

(a) [3 marks] $\lim_{x\to 0} \dfrac{\sin(5x)}{\sin^{-1}(2x)}$

Solution: limit is in the 0/0 form; use L'Hopital's Rule.

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin^{-1}(2x)} = \lim_{x\to 0} \frac{5\cos(5x)}{\frac{2}{\sqrt{1-4x^2}}} = \frac{5}{2}$$

(b) [3 marks] $\lim_{x\to\infty} x\left(\dfrac{\pi}{2} - \arctan x\right)$

Solution: limit is in the $\infty \cdot 0$ form so rewrite it as a fraction, and then use L'Hopital's Rule.

$$\lim_{x\to\infty} x\left(\frac{\pi}{2} - \arctan x\right) = \lim_{x\to\infty} \frac{\left(\frac{\pi}{2} - \arctan x\right)}{1/x} = \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1.$$

(c) [4 marks] $\lim_{x\to\pi/4} 3\,(\tan x)^{\tan(2x)}$

Solution: apart from the factor 3, the limit is in the 1^∞ form, so let the limit be L and then take the natural log of the limit, before using L'Hopital's Rule.

$$\begin{aligned}
\frac{L}{3} = \lim_{x\to\pi/4} (\tan x)^{\tan(2x)} \quad &\Rightarrow \quad \ln(L/3) = \lim_{x\to\pi/4} \ln\left((\tan x)^{\tan(2x)}\right) \\
&\Rightarrow \quad \ln(L/3) = \lim_{x\to\pi/4} \tan(2x)\,\ln(\tan x) \\
&\Rightarrow \quad \ln(L/3) = \lim_{x\to\pi/4} \frac{\ln(\tan x)}{\cot(2x)} \\
&\Rightarrow \quad \ln(L/3) = \lim_{x\to\pi/4} \frac{\frac{\sec^2 x}{\tan x}}{-2\csc^2(2x)} \\
&\Rightarrow \quad \ln(L/3) = \lim_{x\to\pi/4} \frac{2}{-2(-1)^2} = -1 \\
&\Rightarrow \quad L = \frac{3}{e}
\end{aligned}$$

2. [avg: 7.48/10] Find the following derivatives. Simplify your answers.

(a) [4 marks] $F'(2)$ if $F(x) = \displaystyle\int_1^{x^3} \ln(t^2+1)\,dt$

Solution: use the Fundamental Theorem and the chain rule;

$$F'(x) = \ln((x^3)^2+1)\,3x^2 \Rightarrow F'(2) = 3(4)\ \ln(8^2+1) = 12\ln 65$$

(b) [6 marks] $G'(\pi/2)$ if $G(x) = (1+\sin^2 x)^{\cos x}$

Solution: use logarithmic differentiation.

$$\begin{aligned} G(x) = (1+\sin^2 x)^{\cos x} \quad &\Rightarrow \quad \ln(G(x)) = \ln(1+\sin^2 x)^{\cos x} = \cos x\ \ln(1+\sin^2 x) \\ &\Rightarrow \quad \frac{G'(x)}{G(x)} = -\sin x\ \ln(1+\sin^2 x) + \cos x \left(\frac{2\sin x\ \cos x}{1+\sin^2 x}\right) \\ &\Rightarrow \quad G'(x) = G(x)\left(-\sin x\ \ln(1+\sin^2 x) + \cos x \left(\frac{2\sin x\ \cos x}{1+\sin^2 x}\right)\right) \end{aligned}$$

Observe that

$$G(\pi/2) = (1+1)^0 = 1;$$

thus

$$G'(\pi/2) = -\sin(\pi/2)\ln(1+\sin^2(\pi/2) + 0 = -\ln 2.$$

3. [avg: 5.04/10] Find the following:

(a) [2 marks] $\int \left(x^2 + \sin x - e^x\right)\, dx$

Solution: integrate term by term.

$$\int \left(x^2 + \sin x - e^x\right)\, dx = \frac{x^3}{3} - \cos x - e^x + C$$

(b) [4 marks] $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos x}{1 + \sin^2 x}\, dx$

Solution: let $u = \sin x$; then $du = \cos x\, dx$ and

$$\int_{\pi/6}^{\pi/3} \frac{\cos x}{1 + \sin^2 x}\, dx = \int_{1/2}^{\sqrt{3}/2} \frac{1}{1+u^2}\, du = \arctan\left(\frac{\sqrt{3}}{2}\right) - \arctan\left(\frac{1}{2}\right)$$

(c) [4 marks] the average value of $f(x) = \dfrac{x^2+4}{x+2}$ on the interval $[0, 2]$

Solution: the average value is

$$\begin{aligned}
\frac{1}{2}\int_0^2 \frac{x^2+4}{x+2}\, dx &= \frac{1}{2}\int_0^2 \left(x - 2 + \frac{8}{x+2}\right) dx \\
&= \frac{1}{2}\left[\frac{x^2}{2} - 2x + 8\ln(x+2)\right]_0^2 \\
&= \frac{1}{2}\left(2 - 4 + 8\ln(4) - 8\ln 2\right) \\
&= -1 + 4\ln 2
\end{aligned}$$

Alternate Solution: let $u = x + 2$; then $du = dx$, $x = u - 2$, and

$$\begin{aligned}
\frac{1}{2}\int_0^2 \frac{x^2+4}{x+2}\, dx &= \frac{1}{2}\int_2^4 \left(\frac{(u-2)^2 + 4}{u}\right) du \\
&= \frac{1}{2}\int_2^4 \left(\frac{u^2 - 4u + 8}{u}\right) du \\
&= \frac{1}{2}\int_2^4 \left(u - 4 + \frac{8}{u}\right) du \\
&= \frac{1}{2}\left[\frac{u^2}{2} - 4u + 8\ln u\right]_2^4 \\
&= -1 + 4\ln 2, \text{ as before.}
\end{aligned}$$

4. [avg: 5.31/10] Find all the points on the curve with equation $x^2 - 8x + 3y^2 = -8$ at which the tangent line to the curve passes through the origin.

Solution: use implicit differentiation.

$$x^2 - 8x + 3y^2 = -8 \Rightarrow 2x - 8 + 6y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4-x}{3y}.$$

Let (a, b) be a point on the curve at which the tangent line passes through the origin; the equation of the tangent line is

$$y = \frac{4-a}{3b}\,x.$$

Since the point (a, b) is on the curve and on the tangent line we have two equations for a and b :

$$a^2 - 8a + 3b^2 = -8 \quad (1)$$

and

$$b = \left(\frac{4-a}{3b}\right) a \Leftrightarrow 3b^2 = 4a - a^2 \Leftrightarrow a^2 - 4a + 3b^2 = 0. \quad (2)$$

Subtracting equation (1) from equation (2) gives

$$4a = 8 \Leftrightarrow a = 2.$$

Then, from equation (2),

$$3b^2 = 4(2) - 2^2 = 4 \Rightarrow b = \pm\frac{2}{\sqrt{3}}.$$

So the two points on the curve are

$$\left(2, \frac{2}{\sqrt{3}}\right) \text{ and } \left(2, -\frac{2}{\sqrt{3}}\right).$$

5. [avg: 7.14/10] Let $f(x) = x^{7/3} - 7\,x^{1/3}$, for which you may assume

$$f'(x) = \frac{7}{3}x^{4/3} - \frac{7}{3}x^{-2/3} \text{ and } f''(x) = \frac{28}{9}x^{1/3} + \frac{14}{9}x^{-5/3}.$$

(a) [4 marks] Find all the critical points of f and at each critical point determine if f has a maximum or minimum value at the point.

Solution: factor the first derivative: $f'(x) = \dfrac{7(x^2-1)}{3x^{2/3}}$. Since

$$f'(x) = 0 \Rightarrow x = \pm 1; \text{ and } f'(0) \text{ is undefined},$$

the three critical values of f are $x = -1, 0$ or 1. Using the First Derivative test:

$$f'(x) > 0 \text{ if } x < -1 \text{ or } x > 1; \quad \text{and } f'(x) < 0 \text{ if } -1 < x < 1, x \neq 0,$$

the function f has a maximum point at $(-1, 6)$ and a minimum point at $(1, -6)$. There is neither a max nor a min at $(0, 0)$. OR use the Second Derivative test:

$$f''(1) = \frac{14}{3} > 0 \text{ and } f''(-1) = -\frac{14}{3} < 0,$$

so there is a min at $x = 1$ and a max at $x = -1$.

(b) [2 marks] Find all the inflection points of f, if any.

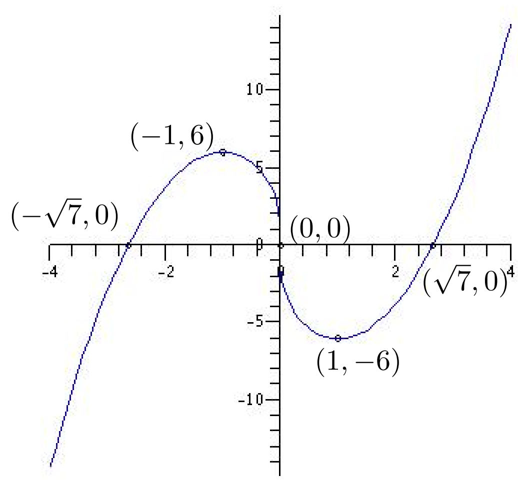
Solution: factor the second derivative: $f''(x) = \dfrac{28x^2 + 14}{9x^{5/3}}$. Since

$$f''(x) > 0 \text{ if } x > 0, \text{ and } f''(x) < 0 \text{ if } x < 0,$$

the only inflection point of f is $(0, 0)$.

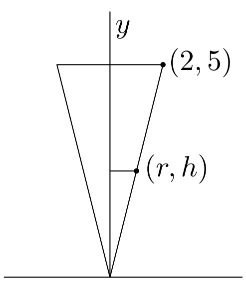
(c) [4 marks] Sketch the graph of f, labeling all critical points, inflection points, and intercepts.

Solution:



6. [avg: 6.26/10] An inverted conical water tank with a height of 5 m and a radius (at the top) of 2 m is drained through a hole in the vertex (at the bottom) at a rate of 1 m^3/min. At what rate is the depth of the water changing when the depth of the water in the tank is 1 m ? Note: the volume of a cone is given by $V = \dfrac{\pi r^2 h}{3}$.

Solution: consider the side view of the tank below (not to scale). Let the depth of the water at time t be h; let the radius of the water surface at time t be r. By similar triangles,



$$\frac{r}{h} = \frac{2}{5} \Leftrightarrow r = \frac{2}{5}h.$$

Thus as a function of h, the volume of the water in the tank at time t is given by

$$V = \frac{\pi r^2 h}{3} = \frac{4\,\pi\,h^3}{75}.$$

Now differentiate implicitly with respect to t :

$$-1 = \frac{dV}{dt} = \frac{4\,\pi\,h^2}{25}\frac{dh}{dt}.$$

So when $h = 1$,

$$\frac{dh}{dt} = -\frac{25}{4\pi}.$$

That is, when the depth of the water in the tank is 1 m, the depth is decreasing at a rate of $\dfrac{25}{4\pi}$ m/min.

7. [avg: 5.17/10]

7.(a) [4 marks] By making use of the Intermediate Value Theorem, find two intervals of the form $[n, n+1]$, where n is an integer, each of which contains a solution to the equation $e^x - 3 - x = 0$.

Solution: let $f(x) = e^x - 3 - x$, which is continuous for all x. Using your calculator you can find that

$$f(0) < 0, \; f(1) < 0, \text{ but } f(2) > 0.$$

By the Intermediate Value Theorem there is a number c in the interval $[1, 2]$ such that $f(c) = 0$.

Similarly,

$$f(-1) < 0, \; f(-2) < 0, \text{ but } f(-3) > 0.$$

By the Intermediate Value Theorem there is a number d in the interval $[-3, , -2]$ such that $f(d) = 0$.

7.(b) [6 marks] Use Newton's method to approximate each of the solutions to the equation $e^x - 3 - x = 0$ correct to 3 decimal places.

Solution: Newton's recursive formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 3 - x_n}{e^{x_n} - 1} = \frac{x_n e^{x_n} - e^{x_n} - 3}{e^{x_n} - 1}, \text{ for } n \geq 0.$$

To find the solution in $[1, 2]$ start with (say) $x_0 = 1.5$. Then

$$x_1 = 1.505259\ldots, \; x_2 = 1.505241\ldots$$

So correct to 3 decimal place the solution is 1.505

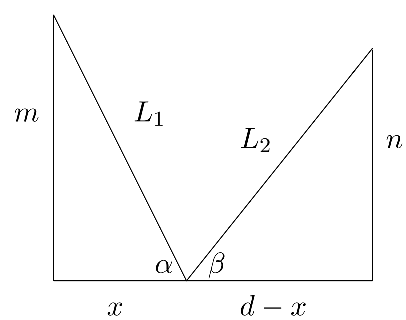
To find the solution in $[-3, -2]$ start with (say) $x_0 = -2.5$. Then

$$x_1 = -2.95528\ldots, \; x_2 = -2.947532\ldots, \; x_3 = -2.947530\ldots$$

so correct to 3 decimal place the solution is -2.947 or -2.948, accept either one, since we aren't going to quibble about rounding up or down if the next digit is 5.

8. [avg: 3.65/10] Two vertical poles of height m and n, respectively, are separated by a horizontal distance d. A rope is stretched from the top of one pole to a point on the level ground between the two poles, and then up to the top of the other pole. To which point on the ground should the rope be attached to minimize the total length of the rope? (For ease of marking: let x be the distance of the point from the base of the pole with height m.)

Solution 1: let the rope be tied to a point on the ground x units from the base of the pole with height m. See the diagram below. Let $L(x)$ be the total length of the rope. Then



$$L(x) = L_1 + L_2 = \sqrt{x^2+m^2} + \sqrt{(d-x)^2+n^2}.$$

The problem is to minimize the value of $L(x)$ on the interval $[0, d]$. The routine way to solve this problem is to find the critical point of L, for which you need

$$L'(x) = \frac{x}{\sqrt{x^2+m^2}} - \frac{d-x}{\sqrt{(d-x)^2+n^2}}.$$

Find the Critical Point:

$$\begin{aligned}
L'(x) = 0 \quad &\Rightarrow \quad \frac{x}{\sqrt{x^2+m^2}} = \frac{d-x}{\sqrt{(d-x)^2+n^2}} \\
&\Rightarrow \quad \frac{x^2}{x^2+m^2} = \frac{(d-x)^2}{(d-x)^2+n^2} \\
&\Rightarrow \quad x^2((d-x)^2+n^2) = (x^2+m^2)(d-x)^2 \\
&\Rightarrow \quad x^2(d-x)^2 + x^2n^2 = x^2(d-x)^2 + m^2(d-x)^2 \\
&\Rightarrow \quad n^2x^2 = m^2(d-x)^2 \\
&\Rightarrow \quad nx = m(d-x) = md - mx \\
&\Rightarrow \quad x(n+m) = md \\
&\Rightarrow \quad x = \frac{md}{n+m}, \text{ and consequently, } d - x = \frac{nd}{m+n}
\end{aligned}$$

Confirm L Has Its Minimum Value at the Critical Point: calculating the second derivative of $L(x)$ is too much work; even using the first derivative test is too much work. Instead, compare the value of $L(x)$ at the critical point with $L(0) = m + \sqrt{d^2+n^2}$ and $L(d) = \sqrt{m^2+d^2} + n$. Note that for the angles α, β as labeled in the diagram, at the critical point

$$\tan\alpha = \frac{m}{x} = \frac{m+n}{d} \text{ and } \tan\beta = \frac{n}{d-x} = \frac{m+n}{d}.$$

So at the critical point, $\alpha = \beta$ and

$$L = L_1 + L_2 = m\csc\alpha + n\csc\alpha = (m+n)\csc\alpha = (m+n)\,\frac{\sqrt{d^2+(m+n)^2}}{m+n} = \sqrt{d^2+(m+n)^2}.$$

Then

$$L < L(0) \Leftrightarrow d^2 + (m+n)^2 < m^2 + 2m\sqrt{n^2+d^2} + n^2 + d^2 \Leftrightarrow n < \sqrt{n^2+d^2},$$

which is true. Similarly, $L < L(d)$. **Aside:** is there an easier way?

Solution 2: the easier way. Flip over one of the poles and then consider the question of the minimum distance from the bottom of one to the top of the other. See the diagram below.

Introducing coordinates, the problem now becomes, minimize the length of the path from the point $(0, -m)$ to the point (d, n), via the point $(x, 0)$. But the path with the minimum length is a straight line. So we must choose x such that the slope of the segment L_1 is the same as the slope of the segment L_2. That is,

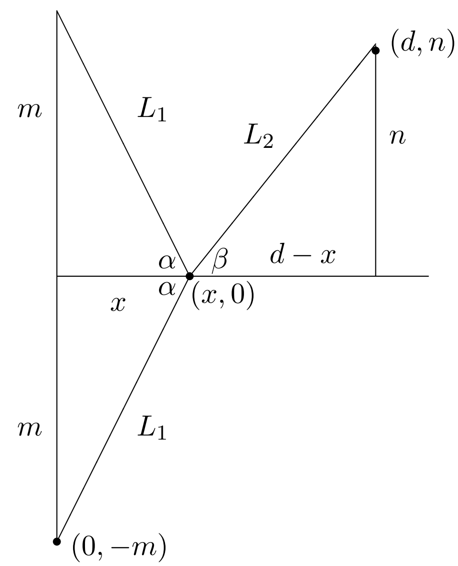
$$\frac{0-(-m)}{x-0} = \frac{n-0}{d-x} \Leftrightarrow \frac{m}{x} = \frac{n}{d-x}.$$

Solving this equation for x gives

$$x = \frac{md}{m+n},$$

as before. **Aside:** from the diagram it is also now obvious why the minimum length of $L = L_1 + L_2$ is

$$L = \sqrt{d^2 + (m+n)^2}.$$



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