MAT186H1F - Calculus I - Fall 2015

Solutions to Term Test 2 - November 24, 2015

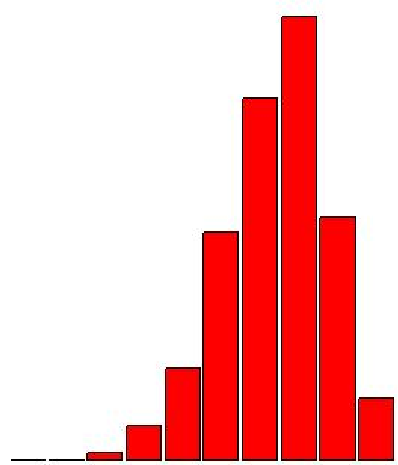
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

1. Each question had a range of 0 to 10. The only question with a failing average was Question 8, although it was both a tutorial and a homework problem. It should not have caught anybody by surprise.
2. Surprisingly, the averages on both Questions 3 and 5 were low, even though these two questions are purely computational! Although, in Question 5(b) things can get really messy if you try to use L'Hopital's Rule more than once.
3. At present, the average mark out of 35, with the first test counting as 15 and the second test as 20, is 25.31/35 or 72.3%.

Breakdown of Results: 879 students wrote this test. The marks ranged from 26.25% to 98.75%, and the average was 68.35%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.2%
A	20.7%	80-89%	16.5%
B	30.1%	70-79%	30.1%
C	24.6%	60-69%	24.6%
D	15.5%	50-59%	15.5%
F	9.1%	40-49%	6.3%
		30-39%	2.4%
		20-29%	0.4%
		10-19%	0.0%
		0-9%	0.0%



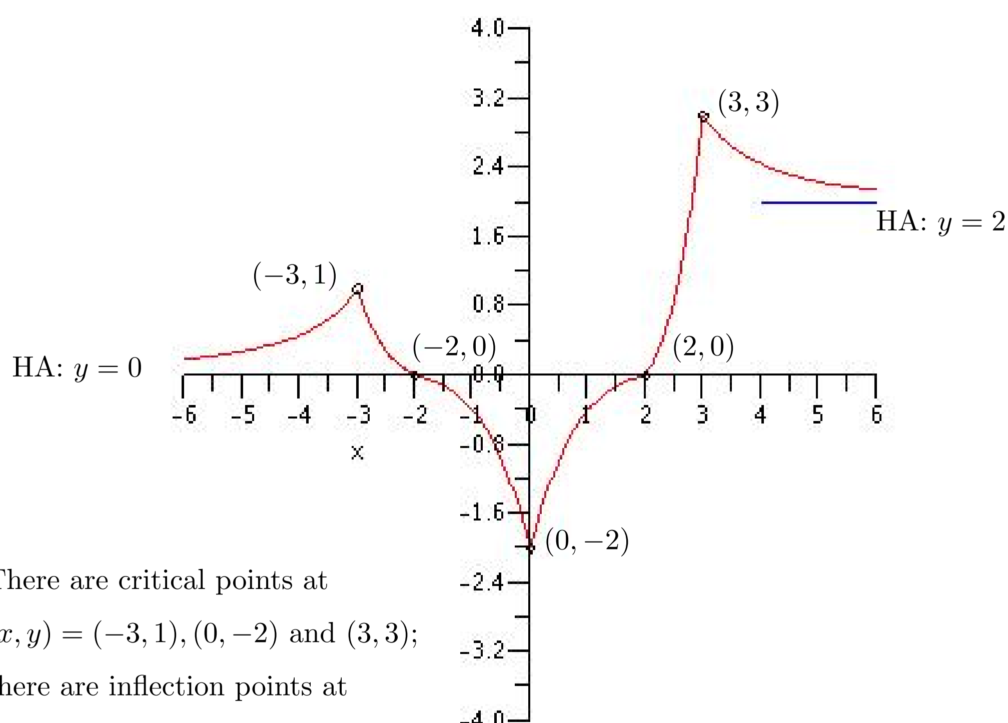
PART I : No explanation is necessary.

1. [avg: 8.15/10] Sketch a possible graph of $y = f(x)$ if f is a function with ALL of the following properties:

   - $f(x)$ is continuous for all x.
   - the only x-intercepts on the graph are $(x, y) = (-2, 0)$ and $(2, 0)$.
   - the only y-intercept on the graph is $(x, y) = (0, -2)$.
   - $f(-3) = 1$ and $f(3) = 3$.
   - $f'(x) > 0$ if $x < -3$ or $0 < x < 3$; $f'(x) < 0$ if $-3 < x < 0$ or $x > 3$.
   - $f''(x) > 0$ if $2 < |x| < 3$ or $|x| > 3$; $f''(x) < 0$ if $0 < |x| < 2$.
   - $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 2$.

   Label all critical points and inflection points on your graph, and all asymptotes to your graph of f.

   Solution: the graph should look something like:



There are critical points at $(x, y) = (-3, 1), (0, -2)$ and $(3, 3)$; there are inflection points at $(x, y) = (-2, 0)$ and $(2, 0)$.

Continued...

2. [avg: 6.20/10] Decide if the following statements are True or False. Each correct choice is worth 1 mark.

(a) If f has a local maximum at c then $f'(c) = 0$. ○ **True** ⊗ **False**

(b) If f has an inflection point at c then $f''(c) = 0$. ○ **True** ⊗ **False**

(c) If $f''(c) = 0$ then f has an inflection point at c. ○ **True** ⊗ **False**

(d) Suppose $f'(x) > 0$ for all x. Then $\lim_{x\to\infty} f(x) = \infty$. ○ **True** ⊗ **False**

(e) If f is continuous for all x then $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$. ⊗ **True** ○ **False**

(f) If F is an antiderivative of f on the interval $[a, b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.
⊗ **True** ○ **False**

(g) If $\int_0^2 h(t)\,dt = 0$ and $\int_1^3 h(t)\,dt = 0$, then $\int_0^3 h(t)\,dt = 0$ ○ **True** ⊗ **False**

(h) If F and G are both antiderivatives of f on $[0, 1]$ and $F(0) = G(0)$, then $F(x) = G(x)$ for all x in $[0, 1]$. ⊗ **True** ○ **False**

(i) If f is continuous on $[a, b]$ then f has an antiderivative on $[a, b]$. ⊗ **True** ○ **False**

(j) Suppose c is a critical point of f such that $f''(c) = 0$. By the Second Derivative Test, c is neither a local maximum nor a local minimum of f. ○ **True** ⊗ **False**

Continued...

PART II : Present **complete** solutions to the following questions in the space provided.

3. [avg: 6.38/10] Find and simplify $\frac{dy}{dx}$ at the point $(x, y) = (2, 2)$ if

   (a) [5 marks] $y = \left(\sin\left(\frac{\pi}{x^2}\right)\right)^{-x}$.

   Solution: use logarithmic differentiation and the product rule, chain rule and quotient rule.

$$\ln y = -x \ln\left(\sin\left(\frac{\pi}{x^2}\right)\right) \Rightarrow \frac{y'}{y} = -\ln\left(\sin\left(\frac{\pi}{x^2}\right)\right) - x \csc\left(\frac{\pi}{x^2}\right) \cos\left(\frac{\pi}{x^2}\right)\left(-\frac{2\pi}{x^3}\right)$$

   Substitute $(x, y) = (2, 2)$:

$$\begin{aligned} \frac{y'}{2} &= -\ln\left(\sin\left(\frac{\pi}{4}\right)\right) - 2\csc\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\left(-\frac{2\pi}{8}\right) \\ \Rightarrow \frac{y'}{2} &= -\ln\left(\frac{1}{\sqrt{2}}\right) + 2\left(\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\pi}{4}\right) = \frac{1}{2}\ln 2 + \frac{\pi}{2} \\ \Rightarrow \frac{dy}{dx} &= \ln 2 + \pi \end{aligned}$$

   (b) [5 marks] $y = x + \int_4^{x^2} \sqrt{t^{3/2} + 1}\, dt$.

   Solution: use the Fundamental Theorem of Calculus, and the chain rule.

$$\frac{dy}{dx} = 1 + 2x\sqrt{(x^2)^{3/2} + 1} = 1 + 2x\sqrt{|x|^3 + 1}.$$

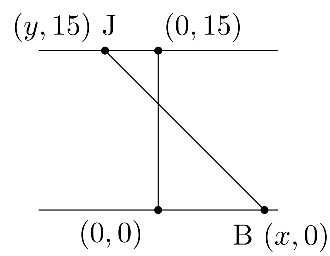
   Substitute $x = 2$:

$$\frac{dy}{dx} = 1 + 2(2)\sqrt{8 + 1} = 1 + 4(3) = 13.$$

Continued...

4. [avg: 7.41/10] Two parallel paths 15 m apart run east-west through the woods. Brooke jogs west to east on one path at 4 m/sec, while Jamail walks east to west on the other path at 2 m/sec. If they pass each other at $t = 0$ how far apart are they 3 seconds later, and how fast is the distance between them changing at that moment?

Solution:



Let Jamail's position at time t be $(y, 15)$ and let Brooke's position at time t be $(x, 0)$ as illustrated on the diagram to the right, where t is measured in seconds since they passed each other. Let D be the distance between them at time t. We have $x = 4t$ and $y = -2t$ and $D^2 = (x - y)^2 + 15^2$. Then:

$$D^2 = (4t - (-2t))^2 + 15^2 = 36t^2 + 225$$

and

$$2D\frac{dD}{dt} = 72t \Leftrightarrow \frac{dD}{dt} = \frac{36t}{D}.$$

So at $t = 3$,

$$D^2 = 36(9) + 225 \Rightarrow D = \sqrt{549} = 3\sqrt{61} \approx 23.43 \text{ m}$$

and

$$\frac{dD}{dt} = \frac{36(3)}{3\sqrt{61}} = \frac{36}{\sqrt{61}} \approx 4.61 \text{ m/sec.}$$

5. [avg: 5.93/10] State which indeterminate form each of the following limits is, and then calculate it.

(a) [5 marks] $\displaystyle\lim_{x\to 0^+}\left(e^{-x}+3x\right)^{2/x}$

Solution: in the 1^∞ form.

$$\begin{aligned}
L &= \lim_{x\to 0^+}\left(e^{-x}+3x\right)^{2/x}\\
\Rightarrow \ln L &= \lim_{x\to 0^+}\frac{2\ln(e^{-x}+3x)}{x}\\
&= 2\lim_{x\to 0^+}\frac{-e^{-x}+3}{e^{-x}+3x},\ \text{by L'Hopital's Rule}\\
&= 2\left(\frac{-1+3}{1+0}\right)=4\\
\Rightarrow L &= e^4
\end{aligned}$$

(b) [5 marks] $\displaystyle\lim_{x\to -\infty}\frac{\pi-2\sec^{-1}(2x)}{\pi+2\tan^{-1}(3x)}$

Solution: in the 0/0 form. Use L'Hopital's Rule.

$$\begin{aligned}
\lim_{x\to -\infty}\frac{\pi-2\sec^{-1}(2x)}{\pi+2\tan^{-1}(3x)} &= \lim_{x\to -\infty}\frac{-2\left(\dfrac{2}{|2x|\sqrt{4x^2-1}}\right)}{2\left(\dfrac{3}{1+9x^2}\right)},\ \text{by L'Hopital's Rule}\\
&= -\frac{1}{3}\lim_{x\to -\infty}\frac{1+9x^2}{|x|^2\sqrt{4-1/x^2}},\ \text{since } \sqrt{x^2}=|x|\\
&= -\frac{1}{3}\lim_{x\to -\infty}\frac{1+9x^2}{x^2\sqrt{4-1/x^2}},\ \text{since } |x|^2=x^2\\
&= -\frac{1}{3}\lim_{x\to -\infty}\frac{1/x^2+9}{\sqrt{4-1/x^2}},\ \text{dividing through by } x^2\\
&= -\frac{1}{3}\left(\frac{9}{2}\right)\\
&= -\frac{3}{2}
\end{aligned}$$

Note: Depending on how you do the algebra, things can get surprisingly messy for such a short question!

Continued...

6. [avg: 8.55/10] A large tank is filled with water when an outflow valve is opened at $t = 0$. Water flows out at a rate, in liters per minute, given by $Q'(t) = 0.1(100 - t^2)$, for $0 \leq t \leq 10$.

(a) [5 marks] Find the amount of water $Q(t)$ that has flowed out of the tank after t minutes, given the initial condition $Q(0) = 0$.

Solution:

$$Q(t) = \int 0.1(100 - t^2)\,dt = \frac{1}{10}\left(100t - \frac{t^3}{3}\right) + C = 10\,t - \frac{t^3}{30} + C$$

To find C use the initial condition $Q(0) = 0$:

$$0 = Q(0) = 0 - 0 + C \Leftrightarrow C = 0.$$

So, in units of liters,

$$Q(t) = 10\,t - \frac{t^3}{30}.$$

(b) [2 marks] How much water has flowed out after 10 minutes?

Solution:

$$Q(10) = 100 - \frac{1000}{30} = \frac{200}{3} \approx 66.7 \text{ liters}$$

(c) [3 marks] How much water flowed out of the tank for $4 \leq t \leq 8$, that is, between the $4th$ and the $8th$ minutes?

Solution:

$$\int_4^8 Q'(t)\,dt = Q(8) - Q(4) = 80 - \frac{8^3}{30} - \left(40 - \frac{4^3}{30}\right) = \frac{376}{15} \approx 25.1 \text{ liters}$$

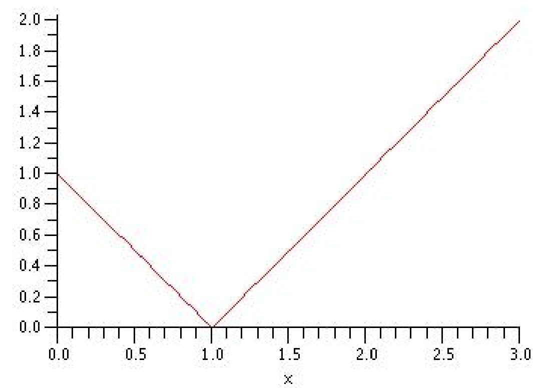
Continued...

7. [avg: 7.28/10] The parts of this question are unrelated. In both parts drawing a graph will be helpful.

(a) [4 marks] Find the value of $\int_0^3 |x-1|\,dx$ geometrically.

Solution: the integral represents the combined area of the two triangles, shown to the right. So

$$\int_0^3 |x-1|\,dx = \frac{1}{2}(1^2) + \frac{1}{2}(2^2) = \frac{5}{2}.$$



Or you can do it using calculus:

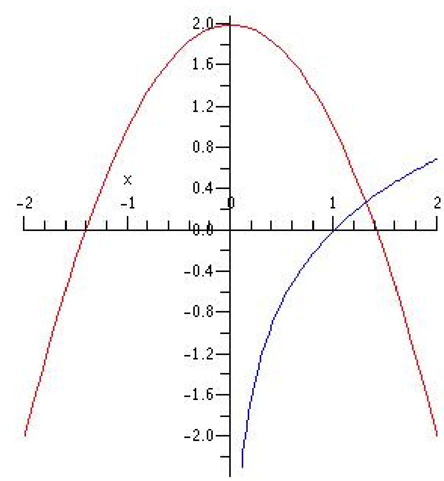
$$\int_0^3 |x-1|\,dx = \int_0^1 (1-x)\,dx + \int_1^3 (x-1)\,dx$$

$$= \left[-\frac{(1-x)^2}{2}\right]_0^1 + \left[\frac{(x-1)^2}{2}\right]_1^3 = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$$

(b) [6 marks] Explain why the equation $2 - x^2 = \ln x$ has exactly one solution, and then approximate it correct to 3 decimal places by defining an appropriate function $f(x)$ and applying Newton's method. (Start with $x_0 = 1$.)

Solution: the parabola with equation $y = 2 - x^2$ intersects the graph of $y = \ln x$ in just one point, somewhere in the interval $[1, 2]$, as the graph to the right shows. Let $f(x) = x^2 + \ln x - 2$, then

$$f'(x) = 2x + \frac{1}{x},$$



and Newton's recursive formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 + \ln x_n - 2}{2x_n + 1/x_n},$$

for $n \geq 0$.

Starting with $x_0 = 1$, and using your calculator, you will find:

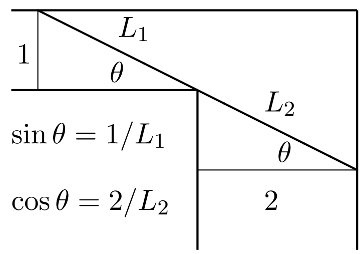
$$x_1 = 1.333333\ldots,\ x_2 = 1.314174353\ldots,\ x_3 = 1.314096806\ldots,\ x_4 = 1.314096804\ldots$$

So correct to three decimal places, the solution is $x = 1.314$

8. [avg: 4.78/10] What is the length of the longest pole that can be carried horizontally around a corner at which a 1-meter corridor and a 2-meter corridor meet at right angles? Draw a diagram!

Solution: in the figure to the right, the length of the pole L is calculated in terms of two pieces, one with length L_1 and one with length L_2. When the pole is touching the corner and the two sides we have

$$L = L_1 + L_2 = \csc\theta + 2\sec\theta,$$



where θ is the angle indicated in the figure. The problem is to find the *shortest* value of L for

$$0 < \theta < \pi/2,$$

since the length of the pole that can be carried around the corner is limited by the tightest fit in the corridor.

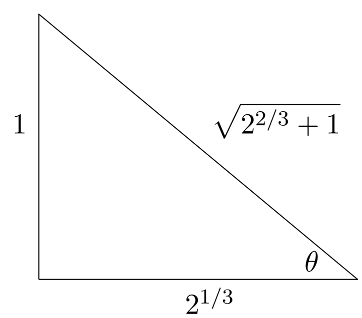
Critical Point:

$$\frac{dL}{d\theta} = 2\sec\theta\tan\theta - \csc\theta\cot\theta;$$

$$\frac{dL}{d\theta} = 0 \Leftrightarrow \frac{2\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\sin^2\theta} = 0 \Leftrightarrow \tan^3\theta = \frac{1}{2} \Leftrightarrow \tan\theta = \frac{1}{2^{1/3}}$$

From the figure to the right, at the critical angle

$$\sec\theta = \frac{\sqrt{2^{2/3}+1}}{2^{1/3}}$$



and

$$\csc\theta = \sqrt{2^{2/3}+1},$$

so the length of the longest pole is

$$L = \sqrt{2^{2/3}+1} + 2\left(\frac{\sqrt{2^{2/3}+1}}{2^{1/3}}\right) = \left(2^{2/3}+1\right)^{3/2} \approx 4.162 \text{ m}.$$

This is actually the shortest value of L, by the second derivative test, since

$$\frac{d^2L}{d\theta^2} = 2\sec\theta\tan^2\theta + 2\sec^3\theta + \csc\theta\cot^2\theta + \csc^3\theta > 0, \text{ for } 0 < \theta < \pi/2.$$

Approximation: its not necessary to find the critical angle, but it is approximately 38.4°.

Continued...

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

Continued...

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

Continued...

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

The end.