MAT186H1F - Calculus I - Fall 2014

Solutions to Term Test 2 - November 11, 2014

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

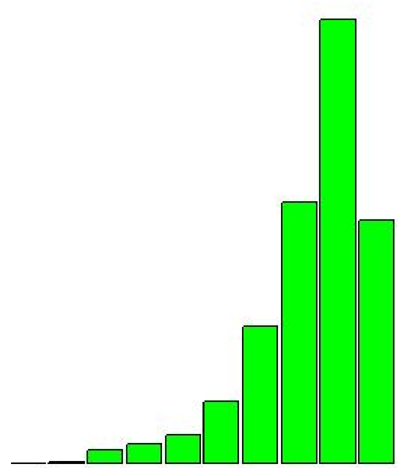
This test consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. The results on this test were very good.
2. Question 7 caused lots of confusion: $L(x) = f(a) + f'(a)(x - a)$ but $f(x) \approx f(a) + f'(a)(x - a)$; so $\sqrt{5} \approx L(5)$. Similarly, in part (b), you should realize that every number your calculator gives you is an approximation, unless it has a finite number of decimals. Many students lost 1 or 2 marks on this question because they confused equal with approximately equal.

Breakdown of Results: 890 students wrote this test. The marks ranged from 18.8% to 100%, and the average was 78%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

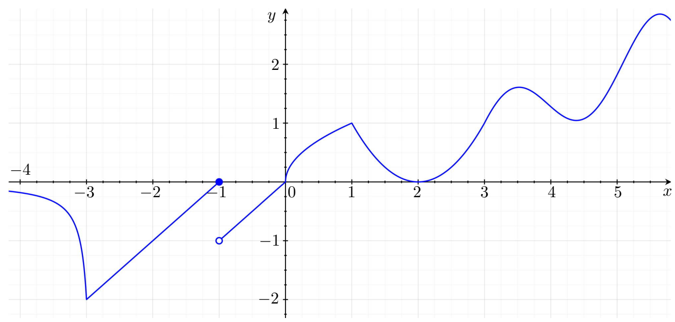
Grade	%	Decade	%
		90-100%	20.1%
A	56.8%	80-89%	36.7%
B	21.6%	70-79%	21.6%
C	11.3%	60-69%	11.3%
D	5.1%	50-59%	5.1%
F	5.2%	40-49%	2.4%
		30-39%	1.6%
		20-29%	1.1%
		10-19%	0.1%
		0-9%	0.0%



PART I : No explanation is necessary.

1. (avg: 6.5/10) The graph of the function f is given to the right. Decide if the following statements about f are True or False.

   Circle your answers.



(a) f is increasing on the interval $(-1, 1)$ True

(b) f is increasing on the interval $(-3, 0)$ False

(c) f has three inflection points. False

(d) f has a local maximum value at $x = -1$ True

(e) f has a vertical tangent line at $x = 0$ False

(f) Excluding $x = -1$, f has seven critical points. True

(g) f has a maximum value at $x = 1$ by the first derivative test. True

(h) f has no absolute minimum value on the interval $(-1, 1)$ True

(i) f has a minimum value at $x = -3$ by the second derivative test. False

(j) f has a minimum value at $x = 2$ by the second derivative test. True

PART II : Present **complete** solutions to the following questions in the space provided.

2. (avg: 8.0/10) Let $f(x) = \dfrac{30\ln x}{x^2}$, for which you may assume $f'(x) = 30\left(\dfrac{1-2\ln x}{x^3}\right)$ and $f''(x) = 30\left(\dfrac{6\ln x - 5}{x^4}\right)$.

   (a) [1 mark] Find the interval(s) on which f is decreasing.

   Solution: for the whole question $x > 0$ since the domain of $\ln x$ is $x > 0$.

   $f'(x) < 0 \Rightarrow 1 - 2\ln x < 0 \Rightarrow \ln x > 1/2 \Rightarrow x > \sqrt{e}$; so f is decreasing on $(\sqrt{e}, \infty)$.

   (b) [1 mark] Find the interval(s) on which f is increasing.

   Solution: $f'(x) > 0 \Rightarrow 1 - 2\ln x > 0 \Rightarrow \ln x < 1/2 \Rightarrow 0 < x < \sqrt{e}$; so f is increasing on $(0, \sqrt{e})$.

   (c) [1 mark] Find the interval(s) on which f is concave up.

   Solution: $f''(x) > 0 \Rightarrow 6\ln x - 5 > 0 \Rightarrow \ln x > 5/6 \Rightarrow x > e^{5/6}$; so f is concave up on $(e^{5/6}, \infty)$.

   (d) [1 mark] Find the interval(s) on which f is concave down.

   Solution:

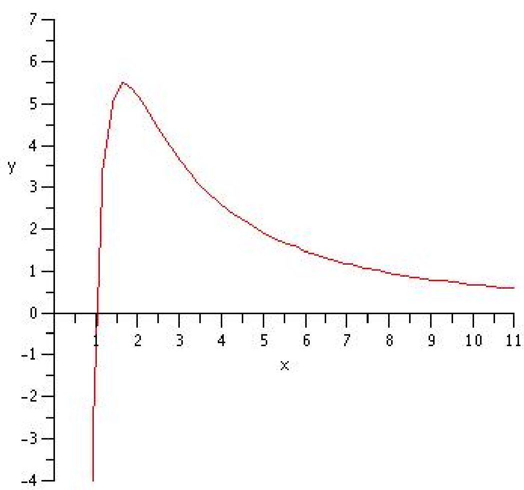
   $f''(x) < 0 \Rightarrow 6\ln x - 5 < 0 \Rightarrow \ln x < 5/6 \Rightarrow x < e^{5/6}$; so f is concave down on $(0, e^{5/6})$.

   (e) [2 marks] Find $\lim\limits_{x\to\infty} f(x)$.

   Solution: use L'Hopital's Rule: $\lim\limits_{x\to\infty} \dfrac{30\ln x}{x^2} = 30 \lim\limits_{x\to\infty} \dfrac{1/x}{2x} = 15 \lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0.$

   (f) [4 marks] Sketch the graph of f labeling all critical points, inflection points and asymptotes.

   Solution:



$x = 0$ is a vertical asymptote;

$y = 0$ is a horizontal asymptote;

$(\sqrt{e}, 15/e)$ is a maximum point;

$(e^{5/6}, 25/e^{5/3})$ is an inflection point.

3. (avg: 9.1/10) Find and simplify $\dfrac{dy}{dx}$ at the point $(x, y) = (1, \pi)$ if

(a) [5 marks] $y \sin y = x^4 - x$

Solution: differentiate implicitly.

$$\frac{dy}{dx}\sin y + y\cos y\frac{dy}{dx} = 4x^3 - 1.$$

At $(x, y) = (1, \pi)$, we have

$$0 - \pi\frac{dy}{dx} = 3 \Leftrightarrow \frac{dy}{dx} = -\frac{3}{\pi}.$$

(b) [5 marks] $y = (4\tan^{-1} x)^x$.

Solution: differentiate logarithmically.

$$\ln y = x\ln(4\tan^{-1} x) \Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(4\tan^{-1} x) + x\left(\frac{4/(1+x^2)}{4\tan^{-1} x}\right).$$

At $(x, y) = (1, \pi)$, we have $\tan^{-1} 1 = \pi/4$ and

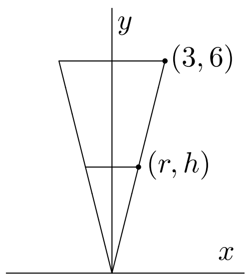
$$\frac{1}{\pi}\frac{dy}{dx} = \ln\pi + \left(\frac{4/2}{\pi}\right).$$

Consequently,

$$\frac{dy}{dx} = \pi\ln\pi + 2.$$

4. (avg: 7.9/10) An inverted conical water tank with a height of 6 m and a radius of 3 m is drained through a hole in the vertex at a rate of 1 m^3/sec. What is the rate of change of the water depth when the water depth is 2 m? (The volume of a conical tank is given by $V = \pi r^2 h/3$.)

Solution: let r be the radius of the surface of the water at depth h above the vertex.



Side view of cone

By similar triangles, $\dfrac{r}{h} = \dfrac{3}{6} \Leftrightarrow r = \dfrac{1}{2}h$. So the volume of the water is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h^2}{4}\right)h = \frac{1}{12}\pi h^3.$$

Now

$$-1 = \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt},$$

so when $h = 2$,

$$\frac{dh}{dt} = -\frac{4}{\pi}\frac{1}{4} = -\frac{1}{\pi};$$

that is the depth of the water is decreasing at a rate of $1/\pi$ meters per second.

5. (avg: 7.6/10) Find the following limits.

(a) [5 marks] $\lim_{x\to 0} \dfrac{e^x - 1 - \sin^{-1} x}{x^3 + 4x^2}$

Solution: this limit is in the 0/0 form so use L'Hopital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1 - \sin^{-1} x}{x^3 + 4x^2} &= \lim_{x\to 0} \frac{e^x - 1/\sqrt{1-x^2}}{3x^2 + 8x} \\
\text{(use L'H again)} &= \lim_{x\to 0} \frac{e^x - x/(1-x^2)^{3/2}}{6x + 8} \\
&= \frac{1}{8}
\end{aligned}$$

(b) [5 marks] $\lim_{x\to\infty} \left(\cos\left(\dfrac{2}{x}\right)\right)^x$

Solution: this limit is in the 1^∞ form, so take the log of the limit, and then use L'Hopital's Rule.

$$\begin{aligned}
L = \lim_{x\to\infty} \left(\cos\left(\frac{2}{x}\right)\right)^x \Rightarrow \ln L &= \lim_{x\to\infty} x \ln\left(\cos\left(\frac{2}{x}\right)\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(\cos\left(2/x\right)\right)}{1/x} \\
\text{(by L'H)} &= \lim_{x\to\infty} \frac{\sec(2/x)(-\sin(2/x)(-2/x^2)}{-1/x^2} \\
&= -2 \lim_{x\to\infty} \tan(2/x) = 0 \\
\Rightarrow L &= e^0 = 1
\end{aligned}$$

Continued...

6. (avg: 8.7/10) Given that the acceleration function of an object moving along a straight line is $a(t) = 3\sin(2t)$, find the position function $s(t)$ if the initial velocity is $v(0) = 1$ and the initial position is $s(0) = 10$.

Solution:

$$v = \int a\,dt = \int 3\sin(2t)\,dt = -\frac{3}{2}\cos(2t) + C.$$

To find C use $v(0) = 1$:

$$1 = -\frac{3}{2}\cos 0 + C \Leftrightarrow C = \frac{5}{2}.$$

Then

$$s = \int v\,dt = \int \left(-\frac{3}{2}\cos(2t) + \frac{5}{2}\right)\,dt = -\frac{3}{4}\sin(2t) + \frac{5}{2}t + D.$$

To find D use $s(0) = 10$:

$$10 = -\frac{3}{4}\sin 0 + 0 + D \Leftrightarrow D = 10.$$

Thus the position function is

$$s(t) = -\frac{3}{4}\sin(2t) + \frac{5}{2}t + 10.$$

Continued...

7. (avg: 7.3/10) (a) [4 marks] Find a linear approximation to $\sqrt{5}$.

Solution: let $f(x) = \sqrt{x}$ and take $a = 4$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$ and the tangent line to f at $a = 4$ has equation

$$y = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4) = 1 + \frac{x}{4}.$$

So to find a linear approximation to $\sqrt{5}$ take $x = 5$; then

$$\sqrt{5} \approx 1 + \frac{5}{4} = 2.25.$$

7.(b) [6 marks] Approximate $\sqrt{5}$ correct to 4 decimals using Newton's method. Hint: let $f(x) = x^2 - 5$.

Solution: let $f(x) = x^2 - 5$. Then $f'(x) = 2x$ and Newton's recursive formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x)} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{x_n^2 + 5}{2x_n},$$

for $n \geq 0$. If you pick $x_0 = 2.25$ (from part (a)) then

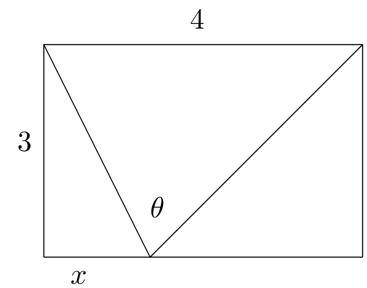
$$x_1 \approx 2.236111111,\ x_2 \approx 2.236067978,\ x_3 \approx 2.23606798,$$

so correct to 4 decimal places,

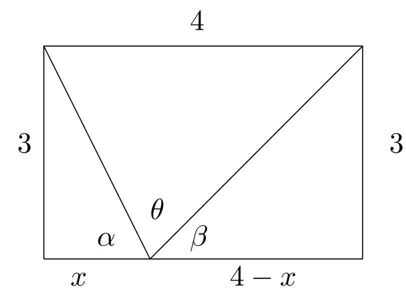
$$\sqrt{5} \approx 2.2361$$

8. (avg: 7.3/10)

Find the value of x that maximizes θ in the figure to the right. Note: there are easy ways and there are hard ways to solve this problem, and in a certain sense the answer is obvious. It's up to you to explain your solution as clearly and concisely as you can.



Solution: let α and β be angles as labeled in the diagram to the right, in which two more lengths have also been included. Since $\alpha + \beta + \theta = \pi$, to maximize θ is the same as minimizing $\alpha + \beta$.



We have:

$$\alpha + \beta = \tan^{-1}\left(\frac{3}{x}\right) + \tan^{-1}\left(\frac{3}{4-x}\right)$$

and

$$\frac{d(\alpha+\beta)}{dx} = \frac{1}{1+(3/x)^2}\left(-\frac{3}{x^2}\right) + \frac{1}{1+(3/(4-x))^2}\left(-\frac{3(-1)}{(4-x)^2}\right) = -\frac{3}{x^2+9} + \frac{3}{(4-x)^2+9}.$$

Then

$$\frac{d(\alpha+\beta)}{dx} = 0 \Rightarrow \frac{3}{x^2+9} = \frac{3}{(4-x)^2+9} \Rightarrow x^2 = (4-x)^2 \Rightarrow x^2 = 16 - 8x + x^2 \Rightarrow x = 2.$$

Finally, confirm that $\alpha + \beta$ is actually at a minimum if $x = 2$, by the second derivative test:

$$\frac{d^2(\alpha+\beta)}{dx^2} = \frac{6x}{(x^2+9)^2} + \frac{6(4-x)}{((4-x)^2+9)^2},$$

which is positive for all $x \in [0, 4]$.

Alternate Solution: If you use the cosine law for Question 8 you will obtain

$$16 = x^2 + 9 + (4-x)^2 + 9 - 2\sqrt{x^2+9}\sqrt{(x-4)^2+9}\,\cos\theta$$

Solving for $\cos\theta$ gives

$$\cos\theta = \frac{x^2-4x+9}{\sqrt{x^2+9}\,\sqrt{(4-x)^2+9}}.$$

Differentiate implicitly to save some work:

$$-\sin\theta\,\frac{d\theta}{dx} = \frac{288(x-2)}{(9+x^2)^{3/2}(25-8x+x^2)^{3/2}},$$

where we have used the quotient rule on the right side, although the details have been omitted.

As before:

$$\frac{d\theta}{dx} = 0 \Rightarrow x = 2.$$

Use the first derivative test to establish that θ is actually maximized at $x = 2$; but first you have to state that $\sin\theta > 0$, since $0 < \theta < \pi$. Then $-\sin\theta < 0$ and:

$$x < 2 \Rightarrow \frac{d\theta}{dx} > 0 \text{ and } x > 2 \Rightarrow \frac{d\theta}{dx} < 0.$$