University of Toronto

SOLUTIONS to MAT 186H1F TERM TEST 2

of **Tuesday, November 3, 2009**

Duration: 90 minutes

TOTAL MARKS: 60

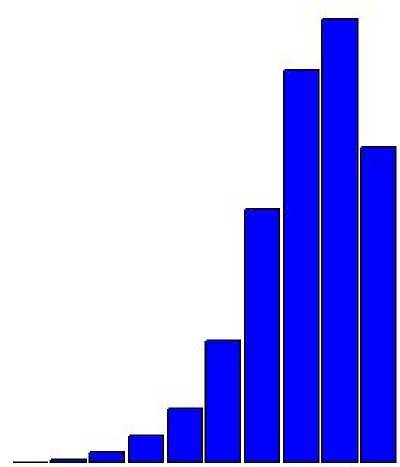
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

- Many students are still throwing away marks by using inappropriate notation or by not identifying what they are calculating.
- Regarding Question 4(b): $|x| > 1$ means $x < -1$ or $x > 1$. Many students just assumed $x > 1$ and ignored the other case. This cost you 2 marks.
- There is an easier way to do Question 4(b): use $\sec^{-1} x = \cos^{-1}(1/x)$ and let $\alpha = \sin^{-1}(1/x)$ and $\beta = \cos^{-1}(1/x)$. Then by basic trig $\alpha + \beta$ is constant, so the derivative must be zero.
- The related rates problem was an example chosen right out of the text book.
- The point of Question 7(a) is to pick a point a for which $a^{-1/3}$ can be calculated *without* a calculator such that a is close to 59. So the only choice is $a = 64$.

Breakdown of Results: 479 students wrote this test. The marks ranged from 10% to 100%, and the average was 75.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	19.4%
A	46.8%	80-89%	27.4%
B	24.2%	70-79%	24.2%
C	15.7%	60-69%	15.7%
D	7.5%	50-59%	7.5%
F	5.8%	40-49%	3.3%
		30-39%	1.7%
		20-29%	0.6%
		10-19%	0.2%
		0-9%	0.0%



1. [8 marks] Find $\frac{dy}{dx}$ if

   (a) [4 marks] $y = e^{x^3} \tan x$.

   Solution: use the product rule.

$$\frac{dy}{dx} = 3x^2 \cdot e^{x^3} \cdot \tan x + e^{x^3} \cdot \sec^2 x$$

   (b) [4 marks] $y = \sqrt{\ln(x^2 + x + 1)}$.

   Solution: use the chain rule, twice.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\ln(x^2 + x + 1)}} \cdot \frac{1}{x^2 + x + 1} \cdot (2x + 1)$$

2. [8 marks] Find all the points on the graph with equation $x^2 + 6x + y^2 + 3 = 0$ such that the tangent line to each of the points passes through the origin.

Solution: let the point of contact be (a, b). Then

$$a^2 + 6a + b^2 + 3 = 0$$

and the slope of the tangent line to the curve at (a, b) is

$$m = \frac{b-0}{a-0} = \frac{b}{a},$$

since the tangent line is to pass through the origin. On the other hand, by implicit differentiation,

$$2x + 6 + 2y\,\frac{dy}{dx} + 0 = 0 \Rightarrow \frac{dy}{dx} = -\frac{x+3}{y}.$$

So the slope of the tangent line to the curve at (a, b) is also given by

$$m = -\frac{a+3}{b}.$$

Hence there are two equations that a and b must satisfy:

$$\frac{b}{a} = -\frac{a+3}{b} \quad\Leftrightarrow\quad b^2 + a^2 = -3a \tag{1}$$

$$a^2 + 6a + b^2 + 3 = 0 \quad\Leftrightarrow\quad a^2 + b^2 = -6a - 3 \tag{2}$$

Comparing equations (1) and (2) implies

$$-3a = -6a - 3 \Leftrightarrow 3a = -3 \Leftrightarrow a = -1.$$

Then, from equaiton (1),

$$b^2 = -3a - a^2 = (-3)(-1) - (-1)^2 = 2 \Rightarrow b = \pm\sqrt{2}.$$

So there are two points on the graph with equation $x^2 + 6x + y^2 + 3 = 0$ from which the tangent lines pass through the origin, namely

$$(-1, \pm\sqrt{2}).$$

3. [9 marks] Find both $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(x, y) = \left(\dfrac{3}{2}, \dfrac{1}{2}\right)$ if $y + \sin(\pi y) = x$.

Solution: differentiate implicitly, with respect to x, twice:

$$\begin{aligned} y + \sin(\pi y) &= x & (3) \\ \Rightarrow y' + \cos(\pi y)\,(\pi y') &= 1 & (4) \\ \Rightarrow y'' - \sin(\pi y)\,(\pi y')\,(\pi y') + \cos(\pi y)(\pi y'') &= 0 & (5) \end{aligned}$$

To find y' at $(x, y) = \left(\dfrac{3}{2}, \dfrac{1}{2}\right)$, substitute $y = \dfrac{1}{2}$ into equation (4) and solve for y' :

$$y' + \cos\left(\frac{\pi}{2}\right)(\pi y') = 1 \Rightarrow y' + 0 = 1 \Rightarrow y' = 1$$

To find y'' at $(x, y) = \left(\dfrac{3}{2}, \dfrac{1}{2}\right)$, substitute $y = \dfrac{1}{2}$ and $y' = 1$ into equation (5) and solve for y'' :

$$\begin{aligned} y'' - \sin\left(\frac{\pi}{2}\right)\left(\pi^2 \cdot (1)^2\right) + \cos\left(\frac{\pi}{2}\right)(\pi\, y'') &= 0 \\ \Rightarrow \quad y'' - \pi^2 + 0 &= 0 \\ \Rightarrow \quad y'' &= \pi^2 \end{aligned}$$

4. [9 marks] Simplify the following derivatives as much as possible.

(a) [4 marks] $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{1-x}{1+x}\right)\right)$

Solution: use the chain rule, and the quotient rule:

$$\begin{aligned}
\frac{d}{dx}\left(\tan^{-1}\left(\frac{1-x}{1+x}\right)\right) &= \frac{1}{1+\left(\dfrac{1-x}{1+x}\right)^2}\frac{d}{dx}\left(\frac{1-x}{1+x}\right) \\
&= \frac{1}{1+\left(\dfrac{1-x}{1+x}\right)^2}\left(\frac{-(1+x)-(1-x)}{(1+x)^2}\right) \\
&= \frac{-2}{(1+x)^2+(1-x)^2} \\
&= \frac{-2}{1+2x+x^2+1-2x+x^2} \\
&= -\frac{1}{1+x^2}
\end{aligned}$$

(b) [5 marks] $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{1}{x}\right)+\sec^{-1}x\right)$, if $|x|>1$.

Solution: more chain rule:

$$\begin{aligned}
&\frac{d}{dx}\left(\sin^{-1}\left(\frac{1}{x}\right)+\sec^{-1}x\right) \\
&= \frac{1}{\sqrt{1-(1/x)^2}}\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{|x|\sqrt{x^2-1}} \\
&= \frac{|x|}{\sqrt{x^2-1}}\left(\frac{-1}{x^2}\right)+\frac{1}{|x|\sqrt{x^2-1}}, \quad \text{since } \sqrt{x^2}=|x| \\
&= -\frac{1}{\sqrt{x^2-1}}\frac{1}{|x|}+\frac{1}{|x|\sqrt{x^2-1}}, \quad \text{since } x^2=|x|^2 \\
&= -\frac{1}{|x|\sqrt{x^2-1}}+\frac{1}{|x|\sqrt{x^2-1}} \\
&= 0
\end{aligned}$$

5. [9 marks] Find the following limits.

(a) [4 marks] $\lim\limits_{x\to 0} \dfrac{\ln(1+x)-x}{e^x-1-x}$

Solution: limit is in the $\dfrac{0}{0}$ form. Use L'Hopital's rule ... twice:

$$\begin{aligned}
\lim_{x\to 0} \frac{\ln(1+x)-x}{e^x-1-x} &= \lim_{x\to 0} \frac{\frac{1}{1+x}-1}{e^x-1} \\
&= \lim_{x\to 0} \frac{\frac{-1}{(1+x)^2}}{e^x} \\
&= \frac{(-1)}{1} \\
&= -1
\end{aligned}$$

(b) [5 marks] $\lim\limits_{x\to 1} (2-x)^{7\tan(\pi x/2)}$

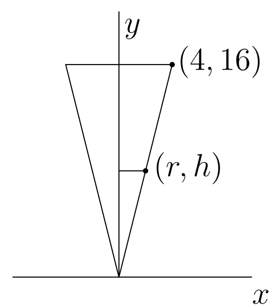
Solution: limit is in the 1^∞ form. Let the limit be L.

$$\begin{aligned}
\ln L &= \lim_{x\to 1} \ln\left((2-x)^{7\tan(\pi x/2)}\right) \\
&= \lim_{x\to 1} 7\tan(\pi x/2)\,\ln(2-x) \\
&= 7\lim_{x\to 1} \frac{\sin(\pi x/2)}{\cos(\pi x/2)}\,\ln(2-x) \\
&= 7\sin(\pi/2)\lim_{x\to 1} \frac{\ln(2-x)}{\cos(\pi x/2)}, \text{ which is in the } \frac{0}{0} \text{ form} \\
&= 7\lim_{x\to 1}\left(\frac{(-1)}{2-x}\right) / \left((-\sin(\pi x/2))(\pi/2)\right), \text{ by L'Hopital's Rule} \\
&= 7\frac{(-1)}{(-\pi/2)} \\
&= \frac{14}{\pi} \\
\Rightarrow L &= e^{14/\pi}
\end{aligned}$$

6. [8 marks] Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top. Suppose also that the liquid is forced out of the cone at a constant rate of 2 cm^3 per min. At what rate is the depth of the liquid in the filter changing when the liquid in the cone is 8 cm deep? (The volume of a cone is given by $V = \frac{\pi}{3} r^2 h$.)

Solution: to the right is a side view of the conical filter. Let h be the depth of the fluid in the filter at time t; let r be the radius of the surface of the fluid at time t. By similar triangles,

$$\frac{r}{h} = \frac{4}{16} \Leftrightarrow r = \frac{h}{4}.$$



Thus the volume of the cone is given by

$$V = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3.$$

Differentiate implicitly with respect to t :

$$\frac{dV}{dt} = \frac{3\pi h^2}{48}\frac{dh}{dt} = \frac{\pi h^2}{16}\frac{dh}{dt}.$$

Use $\dfrac{dV}{dt} = 2$ and let $h = 8$:

$$2 = \frac{\pi(8^2)}{16}\frac{dh}{dt} \Leftrightarrow \frac{dh}{dt} = \frac{1}{2\pi}.$$

So when the liquid in the cone is 8 cm deep, the depth of the liquid is decreasing at at rate of $\frac{1}{2\pi}$ cm/min.

7. [9 marks] The parts of this question are unrelated.

(a) [4 marks] Find the linear approximation of $59^{-1/3}$. Give your answer to 4 decimal places.

Solution: let $f(x) = x^{-1/3}$ and pick $a = 64$. Then $f'(x) = -\dfrac{1}{3}x^{-4/3}$.

$$\begin{aligned} f(x) &\simeq f(a) + f'(a)\,(x-a) \\ \Rightarrow f(59) &\simeq f(64) + f'(64)\,(59-64) \\ &= 64^{-1/3} - \left(\frac{1}{3}\right) 64^{-4/3}\,(-5) \\ &= \frac{1}{4} + \frac{5}{3}\frac{1}{256} \\ &= \frac{197}{768} = 0.256510416\ldots \\ \Rightarrow 59^{-1/3} &\simeq 0.2565, \text{ to 4 decimal places} \end{aligned}$$

(b) [5 marks] Find the value of $\dfrac{dy}{dx}$ at the point $(x, y) = (2, 16)$ if

$$y = x^x\,\sqrt{x^2+12}.$$

Solution: use logarithmic differentiation.

$$\begin{aligned} y = x^x\sqrt{x^2+12} \quad &\Rightarrow \quad \ln y = \ln x^x + \ln\sqrt{x^2+12} \\ &\Rightarrow \quad \ln y = x\ln x + \frac{1}{2}\ln(x^2+12) \\ &\Rightarrow \quad \frac{y'}{y} = \ln x + \frac{x}{x} + \frac{1}{2}\frac{(2x)}{x^2+12} \\ &\Rightarrow \quad y' = y\left(\ln x + 1 + \frac{x}{x^2+12}\right) \end{aligned}$$

At $(x, y) = (2, 16)$,

$$y' = 16\left(\ln 2 + 1 + \frac{2}{4+12}\right) = 16\ln 2 + 18.$$