University of Toronto
Solutions to **MAT186H1S TERM TEST**
of **Wednesday, March 5, 2014**
Duration: 100 minutes

Only aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Instructions: Answer all questions in the booklets provided. Each question is worth 10 marks. If a question has parts, the value of each part is indicated in brackets. **Total Marks: 60**

1. Find $\dfrac{dy}{dx}$ at the point $x = 0$ if

   (a) [4 marks] $y = e^x \sin^{-1} x$

   Solution: by the product rule

$$\frac{dy}{dx} = e^x \sin^{-1} x + \frac{e^x}{\sqrt{1-x^2}};$$

   so at $x = 0$ the derivative has value 1.

   (b) [6 marks] $y = \ln \sqrt{\dfrac{3x+9}{4x+3}}$

   Solution: simplifying gives $y = \dfrac{1}{2}\left(\ln(3x+9) - \ln(4x+3)\right)$, so

$$\frac{dy}{dx} = \frac{1}{2}\frac{3}{3x+9} - \frac{1}{2}\frac{4}{4x+3}.$$

   At $x = 0$ this is equal to $-1/2$.

2. Find the following limits:

(a) [4 marks] $\lim_{x\to 0^+} x^2 \ln x$

Solution: this limit is in the $0 \cdot \infty$ form. Rearrange it as a fraction in the ∞/∞ form and use L'Hopital's Rule:

$$\begin{aligned}
\lim_{x\to 0^+} x^2 \ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \\
&= \lim_{x\to 0^+} \frac{1/x}{-2/x^3} \\
&= -\frac{1}{2} \lim_{x\to 0^+} x^2 \\
&= 0
\end{aligned}$$

(b) [6 marks] $\lim_{x\to 0^+} \left(e^{3x} + x\right)^{2/x}$

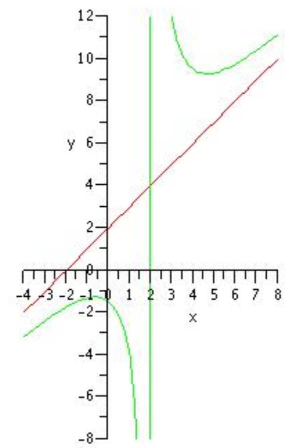
Solution: this limit is in the 1^∞ form, so take the natural log of the limit and then use L'Hoptial's Rule:

$$\begin{aligned}
L = \lim_{x\to 0^+} \left(e^{3x} + x\right)^{2/x} \Rightarrow \ln L &= \lim_{x\to 0^+} \frac{2}{x} \ln\left(e^{3x} + x\right) \\
&= 2 \lim_{x\to 0^+} \frac{3e^{3x} + 1}{e^{3x} + x} \\
&= 2\left(\frac{4}{1}\right) \\
&= 8 \\
\Rightarrow L &= e^8
\end{aligned}$$

3. The parts of this question are not related.

(a) [4 marks] Find all the asymptotes to the graph of $y = \dfrac{x^2+3}{x-2}$.

Solution: by long division $y = \dfrac{x^2+3}{x-2} = x + 2 + \dfrac{7}{x-2}$.



So the graph of y has two asymptotes:
a vertical asymptote $x = 2$,
and a slant asymptote $y = x + 2$.

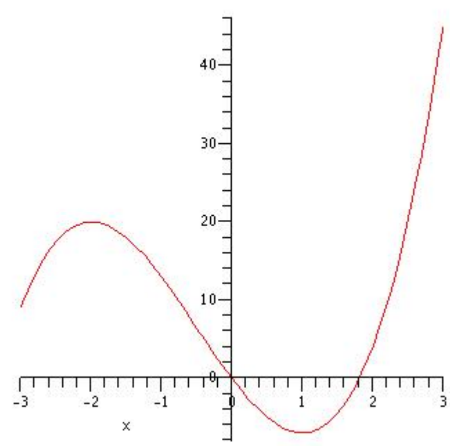
(b) [6 marks] Find the maximum and the minimum value of the function $f(x) = 2x^3 + 3x^2 - 12x$ on the closed interval $[-3, 3]$.

Solution: the extreme values of f will occur at an endpoint of the interval $[-3, 3]$ or at a critical point of f. To find the critical points of f:

$$f'(x) = 6x^2 + 6x - 12 = 6(x-1)(x+2) = 0 \Leftrightarrow x = -2 \text{ or } x = 1.$$

Compare the values of f at $x = -3, -2, 1, 3$:

$$f(-3) = 9$$
$$f(-2) = 20$$
$$f(1) = -7$$
$$f(3) = 45$$



So the maximum value of f on $[-3, 3]$ is $M = 45$ and the minimum value of f on $[-3, 3]$ is $m = -7$.

4. Sketch the graph of $f(x) = x^{4/3} + 4x^{1/3}$, for which you may assume

$$f'(x) = \frac{4}{3}\frac{(x+1)}{x^{2/3}} \text{ and } f''(x) = \frac{4}{9}\frac{(x-2)}{x^{5/3}},$$

labeling all critical points and inflection points, if any.

Solution: $f'(x) > 0 \Leftrightarrow x > -1, x \neq 0$; $f'(x) < 0 \Leftrightarrow x < -1$.

So f is decreasing for $x < -1$ and increasing for $x > -1$.

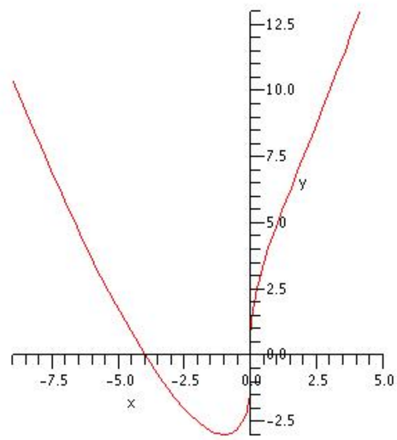
There is a min point at $(x, y) = (-1, -3)$ and a vertical tangent at $x = 0$.

$$f''(x) > 0 \Leftrightarrow x < 0 \text{ or } x > 2;$$

$$f''(x) < 0 \Leftrightarrow 0 < x < 2.$$

So f is concave up for $x < 0$ or $x > 2$ and concave down for $0 < x < 2$. The graph of f has two inflection points:

$$(x, y) = (0, 0) \text{ and } (2, 6(2^{1/3})).$$



5. A cylindrical can, open at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

Solution: let the radius of the can be r, its height h; then its volume V and surface area SA are given by

$$V = \pi r^2 h \text{ and } SA = \pi r^2 + 2\pi r h.$$

Now

$$V = 500 \Rightarrow h = \frac{500}{\pi r^2} \Rightarrow SA = \pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right) = \pi r^2 + \frac{1000}{r}.$$

The problem is: minimize SA for $r > 0$.

Critical Point:

$$\frac{dSA}{dr} = 0 \Rightarrow 2\pi r - \frac{1000}{r^2} = 0 \Rightarrow r = \frac{10}{(2\pi)^{1/3}}.$$

Since

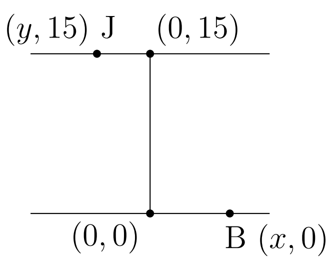
$$\frac{d^2SA}{dr^2} = 2\pi + \frac{2000}{r^3} > 0 \text{ for all } r > 0,$$

SA is indeed minimized at the critical point. The amount of materials will be minimized for

$$r = \frac{10}{(2\pi)^{1/3}} \text{ and } h = \frac{500}{\pi r^2} = \frac{5\,(2^{2/3})}{\pi^{1/3}} = \frac{5\,(2)}{(2\pi)^{1/3}} = r.$$

6. Two parallel paths 15 m apart run east-west through the woods. Brooke jogs west to east on one path at 4 m/sec, while Jamail walks east to west on the other path at 2 m/sec. If they pass each other at $t = 0$ how far apart are they 3 seconds later, and how far is the distance between them changing at that moment?

Solution: Let Jamail's position at time t be $(y, 15)$ and let Brooke's position at time t be $(x, 0)$ as illustrated on the diagram below, where t is measured in seconds since they passed each other. Let D be the distance between them at time t. We have $x = 4t$ and $y = -2t$ and $D^2 = (x - y)^2 + 15^2$. Then:



$$D^2 = (4t - (-2t))^2 + 15^2 = 36t^2 + 225$$

and

$$2D\frac{dD}{dt} = 72t \Leftrightarrow \frac{dD}{dt} = \frac{36t}{D}.$$

So at $t = 3$,

$$D^2 = 36(9) + 225 \Rightarrow D = \sqrt{549} = 3\sqrt{61}$$

and

$$\frac{dD}{dt} = \frac{36(3)}{3\sqrt{61}} = \frac{36}{\sqrt{61}}.$$