University of Toronto<br>Solutions to MAT 186H1S TERM TEST<br>of Wednesday, March 4, 2009<br>Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all seven questions. Present your solutions in the booklets provided. The value for each question is indicated in parentheses beside the question number. Do not use L'Hopital's rule on this test. TOTAL MARKS: 60

## General Comments:

1. The results on this test were much better than last year's results. However, $50 \%$ of the class is below $60 \%$; for these students the quiz marks become extremely important.
2. Questions 3 and 4 were right out of the homework; these questions should have been aced by all.
3. Question 1 was almost a carbon copy of last year's question 1 ; yet many students did very poorly on it.
4. Question 6 was identical to question 4 of the first MAT186H1F test in October, 2008.

Breakdown of Results: 57 students wrote this test. The marks ranged from $30 \%$ to $90 \%$, and the average was $60.7 \%$. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | $\%$ | Decade | $\%$ |
| ---: | :--- | ---: | :--- |
|  |  | $90-100 \%$ | $1.7 \%$ |
| A | $14.0 \%$ | $80-89 \%$ | $12.3 \%$ |
| B | $14.0 \%$ | $70-79 \%$ | $14.0 \%$ |
| C | $21.1 \%$ | $60-69 \%$ | $21.1 \%$ |
| D | $28.1 \%$ | $50-59 \%$ | $28.1 \%$ |
| F | $22.8 \%$ | $40-49 \%$ | $19.3 \%$ |
|  |  | $30-39 \%$ | $3.5 \%$ |
|  |  | $20-29 \%$ | $0.0 \%$ |
|  |  | $10-19 \%$ | $0.0 \%$ |
|  |  | $0-9 \%$ | $0.0 \%$ |



1. [9 marks] Assume $\cos x=\frac{1}{3}$ and $\sin x<0$. Find the exact values of the following:
(a) $[2$ marks $] \sin x$

## Solution:

$$
\begin{aligned}
\sin x & =-\sqrt{1-\cos ^{2} x} \\
& =-\sqrt{1-\left(\frac{1}{3}\right)^{2}} \\
& =-\sqrt{1-\left(\frac{1}{9}\right)} \\
& =-\sqrt{\frac{8}{9}} \\
& =-\frac{2 \sqrt{2}}{3}
\end{aligned}
$$

(b) $[4$ marks $] \cos \left(x-\frac{\pi}{6}\right)$

## Solution:

$$
\begin{aligned}
\cos \left(x-\frac{\pi}{6}\right) & =\cos x \cos \frac{\pi}{6}+\sin x \sin \frac{\pi}{6} \\
& =\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(-\frac{2 \sqrt{2}}{3}\right)\left(\frac{1}{2}\right) \\
& =\frac{\sqrt{3}-2 \sqrt{2}}{6}
\end{aligned}
$$

(c) $[3$ marks $] \cos (2 x)$

## Solution:

$$
\begin{aligned}
\cos (2 x) & =2 \cos ^{2} x-1 \\
& =2\left(\frac{1}{3}\right)^{2}-1 \\
& =-\frac{7}{9}
\end{aligned}
$$

2. [12 marks] State the Intermediate Value Property and use it to explain why the equation $e^{x}-2+x^{2}=0$ has at least two solutions in the interval $[-2,2]$. Then approximate any one of the solutions to the equation $e^{x}-2+x^{2}=0$ correct to three decimal places by using Newton's method.

Intermediate Value Property: Suppose that the function $f$ is continuous on the closed interval $[a, b]$. Then $f(x)$ assumes every intermediate value between $f(a)$ and $f(b)$. That is, if $K$ is any number between $f(a)$ and $f(b)$, there exists at least one number $c$ in $(a, b)$ such that $f(c)=K$. (as stated on page of 97 of text book)

Solution: Let $f(x)=e^{x}-2+x^{2}$, which is a continuous function for all $x$. Observe that

$$
f(-2)=e^{-2}+2>0 \text { and } f(0)=-1<0 .
$$

So by the Intermediate Value Property, there is a number $c_{1} \in(-2,0)$ such that

$$
f\left(c_{1}\right)=0
$$

Similarly,

$$
f(0)=-1<0 \text { and } f(2)=e^{2}+2>0
$$

so by the Intermediate Value Property, there is a number $c_{2} \in(0,2)$ such that

$$
f\left(c_{2}\right)=0
$$

Thus the equation $f(x)=0$ has at least two solutions in the interval $[-2,2]$.

## Newton's Method:

$$
f(x)=e^{x}-2+x^{2} \text { and } f^{\prime}(x)=e^{x}+2 x
$$

so the recursive formula for Newton's method is

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=x_{n}-\frac{e^{x_{n}}-2+x_{n}^{2}}{e^{x_{n}}+2 x_{n}}=\frac{x_{n} e^{x_{n}}+x_{n}^{2}-e^{x_{n}}+2}{e^{x_{n}}+2 x_{n}} .
$$

Either of the following calculations will do:

$$
\begin{gathered}
x_{0}=1 \Rightarrow x_{1}=0.63582 \ldots \Rightarrow x_{2}=0.54315 \ldots \Rightarrow x_{3}=0.53729 \ldots \Rightarrow x_{4}=0.53727 \ldots \\
x_{0}=-1 \Rightarrow x_{1}=-1.38730 \ldots \Rightarrow x_{2}=-1.31824 \ldots \Rightarrow x_{3}=-1.31597 \ldots \Rightarrow x_{4}=-1.31597 \ldots
\end{gathered}
$$

So correct to three decimal places the solutions are 0.537 or -1.315 .
3. [7 marks] Find the open intervals on which the graph of the function

$$
f(x)=10 x^{2 / 3}+2 x^{5 / 3}
$$

is increasing, and the open intervals on which it is decreasing.

## Solution:

$$
f^{\prime}(x)=\frac{20}{3} x^{-1 / 3}+\frac{10}{3} x^{2 / 3}=\frac{10(2+x)}{3 x^{1 / 3}}
$$

The critical points of $f$ are $x=-2$ and $x=0$. Use test points on intervals to find that

$$
\begin{gathered}
f^{\prime}(-27)=\frac{-250}{-9}=\frac{250}{9}>0 \\
f^{\prime}(-1)=\frac{10}{-3}=-\frac{10}{3}<0
\end{gathered}
$$

and

$$
f^{\prime}(8)=\frac{100}{6}=\frac{50}{3}>0
$$

So

|  | ${ }^{-2}$ | 0 |
| :--- | :--- | :--- |
| $f^{\prime}(x)>0$ |  | $f^{\prime}(x)<0$ |

That is, $f$ is increasing on $(-\infty,-2)$ and $(0, \infty)$; and $f$ is decreasing on $(-2,0)$.
4. [8 marks] Sand is being emptied from a hopper at a rate of $1 \mathrm{~m}^{3} / \mathrm{sec}$. The sand forms a conical pile whose height is always twice its radius at the base. At what rate is the radius at the base of the pile increasing when its height is 2 m ? (The volume of a cone is given by $V=\frac{1}{3} \pi r^{2} h$.)

## Solution:

$$
\begin{aligned}
h=2 r & \Rightarrow V=\frac{1}{3} \pi r^{2}(2 r) \\
& \Rightarrow V=\frac{2}{3} \pi r^{3} \\
& \Rightarrow \frac{d V}{d t}=2 \pi r^{2} \frac{d r}{d t}
\end{aligned}
$$



Problem is: given $\frac{d V}{d t}=1$ find $\frac{d r}{d t}$ when $h=2 \Leftrightarrow r=1$.
Substitute known values and solve for $\frac{d r}{d t}$ :

$$
\begin{aligned}
1 & =2 \pi(1)^{2} \frac{d r}{d t} \\
\Rightarrow \frac{d r}{d t} & =\frac{1}{2 \pi}
\end{aligned}
$$

So the radius of the sand pile is increasing at $\frac{1}{2 \pi} \mathrm{~m} / \mathrm{sec}$ when its height is 2 m .
5. [8 marks] A rectangular box has a square base with edges at least 1 cm long. Its total surface area is $600 \mathrm{~cm}^{2}$. What is the largest possible volume such a box can have?

Solution: let $x \times x$ be dimensions of the square base; let $y$ be the height of the box.

Use

$$
V=x^{2} y
$$

and

$$
S A=2 x^{2}+4 x y
$$

Then

$$
\begin{aligned}
S A=600 & \Leftrightarrow 2 x^{2}+4 x y=600 \\
& \Leftrightarrow y=\frac{150}{x}-\frac{x}{2} \\
& \Rightarrow V=x^{2}\left(\frac{150}{x}-\frac{x}{2}\right) \\
& \Rightarrow V=150 x-\frac{x^{3}}{2}
\end{aligned}
$$



Problem is: find maximum value of $V=150 x-\frac{x^{3}}{2}$ on the interval $[1, \infty)$.
Critical Point:

$$
\frac{d V}{d x}=150-\frac{3}{2} x^{2}=0 \Rightarrow x=10
$$

## First Derivative Test:

$$
\frac{d V}{d x}>0 \text { if } 1 \leq x<10, \quad \frac{d V}{d x}<0 \text { if } x>10
$$

thus $V$ has a maximum value at $x=10$, by the first derivative test.
At the critical point, $x=10, y=10$ and $V=10^{3}=1000$. So the maximum volume of the box is $1000 \mathrm{~cm}^{3}$.

## Second Derivative Test:

$$
\frac{d^{2} V}{d x^{2}}=-3 x<0 \text { if } x \geq 1
$$

Thus, as before, at the critical point, $x=10, y=10, V=1000$, the volume is a maximum.
6. [9 marks] Find the following limits.
(a) [5 marks] $\lim _{z \rightarrow 0} \frac{\tan (5 z)}{\sin (3 z)}$

Solution: Make use of the basic trig limit $\lim _{h \rightarrow 0} \frac{\sin h}{h}=1$.

$$
\begin{aligned}
\lim _{z \rightarrow 0} \frac{\tan (5 z)}{\sin (3 z)} & =\lim _{z \rightarrow 0}\left(\frac{1}{\cos (5 z)} \frac{\sin (5 z)}{z} \frac{z}{\sin (3 z)}\right) \\
& =\frac{1}{1} \cdot \frac{5}{3} \cdot \lim _{z \rightarrow 0} \frac{\sin (5 z)}{(5 z)} \cdot \lim _{z \rightarrow 0} \frac{(3 z)}{\sin (3 z)} \\
& =\frac{5}{3} \cdot 1 \cdot \frac{1}{1} \\
& =\frac{5}{3}
\end{aligned}
$$

(b) [4 marks] $\lim _{x \rightarrow 0} x^{2} \sin \left(1+\frac{1}{x}+\frac{1}{x^{2}}\right)$

Solution: Use the Squeeze Law.

$$
\begin{aligned}
x \neq 0 & \Rightarrow-1 \leq \sin \left(1+\frac{1}{x}+\frac{1}{x^{2}}\right) \leq 1 \\
& \Rightarrow-x^{2} \leq x^{2} \sin \left(1+\frac{1}{x}+\frac{1}{x^{2}}\right) \leq x^{2}
\end{aligned}
$$

Both

$$
\lim _{x \rightarrow 0}\left(-x^{2}\right)=0 \text { and } \lim _{x \rightarrow 0} x^{2}=0
$$

so by the Squeeze Law,

$$
\lim _{x \rightarrow 0} x^{2} \sin \left(1+\frac{1}{x}+\frac{1}{x^{2}}\right)=0
$$

as well.
7. [7 marks] Find the value of $\frac{d y}{d x}$ at the point $(x, y)=(2,1)$ if

$$
y^{2}+1=x^{\sin (y \pi / x)}
$$

Solution: differentiate logarithmically.

$$
\begin{aligned}
y^{2}+1=x^{\sin (y \pi / x)} & \Rightarrow \ln \left(y^{2}+1\right)=\ln \left(x^{\sin (y \pi / x)}\right)=\sin (y \pi / x) \ln x \\
& \Rightarrow \frac{2 y y^{\prime}}{y^{2}+1}=\cos (y \pi / x)\left(\frac{\pi y^{\prime} x-\pi y}{x^{2}}\right) \ln x+\sin (y \pi / x) \frac{1}{x} \\
x=2, y=1 & \Rightarrow \frac{2 y^{\prime}}{1+1}=\cos (\pi / 2)\left(\frac{2 \pi y^{\prime}-\pi}{4}\right) \ln 2+\sin (\pi / 2)\left(\frac{1}{2}\right) \\
& \Rightarrow y^{\prime}=0+1\left(\frac{1}{2}\right) \\
& \Rightarrow y^{\prime}=\frac{1}{2}
\end{aligned}
$$

