University of Toronto
Solutions to **MAT 186H1S TERM TEST**
of **Thursday, March 6, 2008**
Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parantheses beside the question number.

TOTAL MARKS: 60

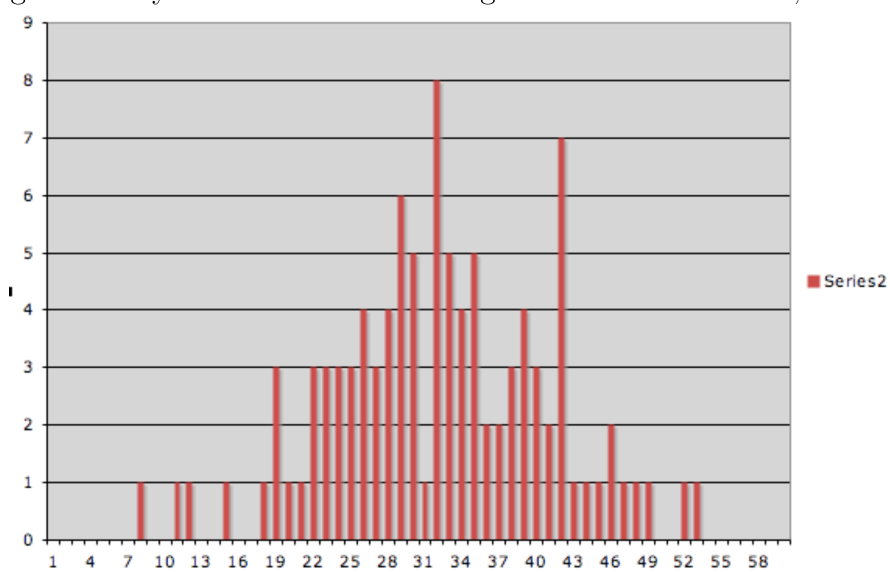
NAME: ______________________

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

LOCATION: (eg HA403) ______________________

Results: 100 students wrote this test. The marks ranged from 13% to 88%, and the average was only 53.8%. Here is a histogram of the test marks, out of 60.



1. [8 marks] Suppose $\sin x = -\dfrac{2}{3}$ and $\cos x < 0$. Find the exact values of the following:

(a) [2 marks] $\cos x$

Solution:

$$\begin{aligned}
\cos x &= -\sqrt{1-\sin^2 x} \\
&= -\sqrt{1-\left(-\frac{2}{3}\right)^2} \\
&= -\sqrt{\frac{5}{9}} \\
&= -\frac{\sqrt{5}}{3}
\end{aligned}$$

(b) [3 marks] $\sin\left(x+\dfrac{\pi}{3}\right)$

Solution:

$$\begin{aligned}
\sin\left(x+\frac{\pi}{3}\right) &= \sin x\cos(\pi/3)+\cos x\sin(\pi/3) \\
&= \left(-\frac{2}{3}\right)\left(\frac{1}{2}\right)+\left(-\frac{\sqrt{5}}{3}\right)\left(\frac{\sqrt{3}}{2}\right) \\
&= -\frac{2+\sqrt{15}}{6}
\end{aligned}$$

(c) [3 marks] $\cos(2x)$

Solution:

$$\begin{aligned}
\cos(2x) &= 1-2\sin^2 x \\
&= 1-2\left(\frac{4}{9}\right) \\
&= \frac{1}{9}
\end{aligned}$$

2. [8 marks] Find the following limits.

(a) [3 marks] $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

Solution:

$$\begin{aligned} \lim_{x \to 2} \frac{x^3 - 8}{x - 2} &= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2} \\ &= \lim_{x \to 2} (x^2 + 2x + 4) \\ &= 12 \end{aligned}$$

(b) [3 marks] $\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1}$

Solution:

$$\begin{aligned} \lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1} &= \lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1} \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2} \\ &= \lim_{x \to 1} \frac{x + 3 - 4}{(x-1)(\sqrt{x+3} + 2)} \\ &= \lim_{x \to 1} \frac{1}{\sqrt{x+3} + 2} \\ &= \frac{1}{4} \end{aligned}$$

(c) [2 marks] $\lim_{x \to 0} x^2 \sin(-1/x)$

Solution: use the squeeze law. For $x \neq 0$,

$$\begin{aligned} & -1 \leq \sin(-1/x) \leq 1 \\ \Rightarrow\ & -x^2 \leq x^2 \sin(-1/x) \leq x^2 \\ \Rightarrow\ & 0 = \lim_{x \to 0} (-x^2) \leq \lim_{x \to 0} x^2 \sin(-1/x) \leq \lim_{x \to 0} x^2 = 0 \\ \Rightarrow\ & \lim_{x \to 0} x^2 \sin(-1/x) = 0 \end{aligned}$$

3. [7 marks] Find the value of $\frac{dy}{dx}$ at the point $(x, y) = (1, 1)$ if

(a) [3 marks] $y + 1 = 2\,e^{x-y}$

Solution: Differentiate implicitly.

$$\begin{aligned} y + 1 = 2\,e^{x-y} \quad &\Rightarrow \quad y' = 2e^{x-y}(1 - y') \\ \text{sub } x = 1, y = 1 \quad &\Rightarrow \quad y' = 2(1 - y') \\ &\Rightarrow \quad 3y' = 2 \\ &\Rightarrow \quad y' = \frac{2}{3} \end{aligned}$$

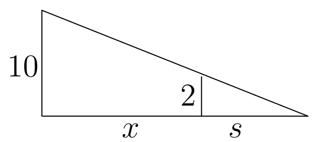
(b) [4 marks] $y + 1 = (x + x^2)^y$

Solution: Differentiate logarithmically.

$$\begin{aligned} y + 1 = (x + x^2)^y \quad &\Rightarrow \quad \ln(y + 1) = y \ln(x + x^2) \\ &\Rightarrow \quad \frac{y'}{y + 1} = y' \ln(x + x^2) + y\frac{1 + 2x}{x + x^2} \\ \text{sub } x = 1, y = 1 \quad &\Rightarrow \quad \frac{y'}{2} = y' \ln 2 + \frac{3}{2} \\ &\Rightarrow \quad y' = \frac{3}{1 - 2\ln 2} \end{aligned}$$

4. [7 marks] A pedestrian walks (on a level sidewalk) away from a lamppost that is 10 metres tall. The pedestrian walks at a constant rate of 1 metre per sec, and is 2 metres tall . At what rate is the pedestrian's shadow increasing when the pedestrian is 15 metres from the base of the lamppost?

Solution:



By similar triangles:

$$\frac{s}{2} = \frac{s+x}{10} \Rightarrow x = 4s.$$

So

$$\frac{ds}{dt} = \frac{1}{4}\frac{dx}{dt}.$$

Thus when $x = 15$, the pedestrian's shadow is increasing at the rate of 0.25 m/sec.

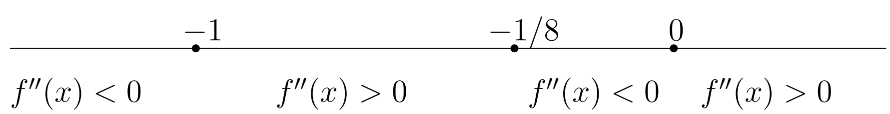
5. [16 marks] This question has four parts and covers two pages. Let $f(x) = \dfrac{x^{1/3}+4}{x^{1/3}+1}$; for which

$$f'(x) = -\frac{1}{x^{2/3}\,(x^{1/3}+1)^2} \quad \text{and} \quad f''(x) = \frac{2}{3}\,\frac{2x^{1/3}+1}{x^{5/3}\,(x^{1/3}+1)^3}.$$

(a) [4 marks] On the following number line, label all critical points of f, if any, all discontinuities of f, if any; and indicate the intervals on which f is increasing, or decreasing.

So f is decreasing on $(-\infty, -1), (-1, 0)$ and $(0, \infty)$; f is never increasing.

(b) [4 marks] On the following number line, label all inflection points of f, if any, all discontinuities of f, if any; and indicate the intervals on which f is concave up, or concave down.



So f is concave down on $(-\infty, -1)$ and $(-1/8, 0)$; f is concave up on $(-1, -1/8)$ and $(0, \infty)$.

(c) [4 marks] Find all the horizontal and vertical asymptotes to the graph of f, if any.

Solution: Vertical asymptote at $x = -1$ since

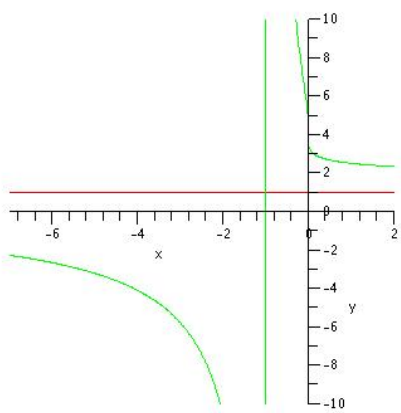
$$\lim_{x\to -1^+} \frac{x^{1/3}+4}{x^{1/3}+1} = \infty; \lim_{x\to -1^-} \frac{x^{1/3}+4}{x^{1/3}+1} = -\infty$$

Horizontal asymptote at $y = 1$ since

$$\lim_{x\to \pm\infty} \frac{x^{1/3}+4}{x^{1/3}+1} = \lim_{x\to \pm\infty} \frac{1+4/x^{1/3}}{1+1/x^{1/3}} = \frac{1+0}{1+0} = 1$$

(d) [4 marks] Plot the graph of $y = f(x)$, labeling all critical points, all inflection points, and all asymptotes, if any.

Graph:



From Maple; not a very good picture.

Inflection points at:

$$(-1/8, 7) \text{ and } (0, 4).$$

6. [7 marks] Let

$$f(x) = \frac{\sin x}{2x^2 - 3\pi x}.$$

Find all points where the function is not continuos, and at each such point $x = a$, calculate both

$$\lim_{x\to a^-} f(x) \text{ and } \lim_{x\to a^+} f(x),$$

or explain why they do not exist.

Solution:

$$2x^2 - 3\pi x = x(2x - 3\pi) = 0 \Leftrightarrow x = 0 \text{ or } x = \frac{3}{2}\pi.$$

So the discontinuities of f are at $x = 0$ and $x = 3\pi/2$,

At $a = 0:$

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{\sin x}{2x^2 - 3\pi x} \\
&= \lim_{x\to 0} \frac{\sin x}{x(x - 3\pi)} \\
&= -\frac{1}{3\pi} \lim_{x\to 0} \frac{\sin x}{x} \\
&= -\frac{1}{3\pi} \cdot 1 \text{ (by basic trig limit)} \\
&= -\frac{1}{3\pi}
\end{aligned}$$

So both

$$\lim_{x\to 0^-} f(x) = -\frac{1}{3\pi} \text{ and } \lim_{x\to 0^+} f(x) = -\frac{1}{3\pi}.$$

At $a = 3\pi/2 : \sin(3\pi/2) = -1$, so

$$\lim_{x\to 3\pi/2^-} f(x) = \lim_{x\to 3\pi/2^-} \frac{\sin x}{x} \frac{1}{2x - 3\pi} = \infty$$

and

$$\lim_{x\to 3\pi/2^+} f(x) = \lim_{x\to 3\pi/2^+} \frac{\sin x}{x} \frac{1}{2x - 3\pi} = -\infty.$$

7. [7 marks]

(a) [3 marks] Find an approximation to $25^{2/3}$ by using the linear approximation of $f(x) = x^{2/3}$ at $a = 27$. (Express your answer to five decimal places.)

Solution:

$$f'(x) = \frac{2}{3}x^{-1/3};\, f(27) = 9;\, f'(27) = \frac{2}{9},$$

so the tangent line to f at $a = 27$ is

$$y = 9 + \frac{2}{9}(x - 27).$$

Hence

$$25^{2/3} = f(25) \simeq 9 + \frac{2}{9}(25 - 27) = \frac{77}{9} = 8.55555\ldots.$$

(b) [4 marks] Find an approximation to $25^{2/3}$ by applying Newton's method to the equation

$$x^{3/2} - 25 = 0;$$

start with $x_0 = 9$ and compute x_1 and x_2. (Express your answers to five decimal places.)

Soltuion:

$$f(x) = x^{3/2} - 25 \Rightarrow f'(x) = \frac{3}{2}\sqrt{x};$$

so Newton's iterative formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{3/2} - 25}{3\sqrt{x_n}/2} \\ &= \frac{1}{3}\left(x_n + \frac{50}{\sqrt{x_n}}\right) \end{aligned}$$

Then:

$$x_0 = 9 \Rightarrow x_1 = \frac{77}{9} = 8.55555\ldots.$$

and

$$x_1 = \frac{77}{9} \Rightarrow x_2 = \frac{77}{27} + \frac{50}{\sqrt{77}} = 8.54988$$