University of Toronto
Faculty of Applied Science and Engineering
Solutions To Final Examination, December 2016
Duration: 2 and $1 / 2$ hrs
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS
MAT186H1F - Calculus I
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Exam Type: A.
Aids permitted: Casio FX-991 or Sharp EL-520 calculator. This exam consists of 9 questions. Each question is worth 10 marks.

Total Marks: 90

Breakdown of Results: 748 students wrote this exam. The marks ranged from $5.6 \%$ to $100 \%$, and the average was $69.9 \%$. There were four perfect papers. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | $\%$ | Decade | $\%$ |
| ---: | :--- | ---: | :--- |
|  |  | $90-100 \%$ | $12.7 \%$ |
| A | $34.4 \%$ | $80-89 \%$ | $21.7 \%$ |
| B | $23.8 \%$ | $70-79 \%$ | $23.8 \%$ |
| C | $15.9 \%$ | $60-69 \%$ | $15.9 \%$ |
| D | $12.4 \%$ | $50-59 \%$ | $12.4 \%$ |
| F | $13.4 \%$ | $40-49 \%$ | $8.2 \%$ |
|  |  | $30-39 \%$ | $2.7 \%$ |
|  |  | $20-29 \%$ | $1.3 \%$ |
|  |  | $10-19 \%$ | $0.9 \%$ |
|  |  | $0-9 \%$ | $0.3 \%$ |



1. [avg: 7.48/10] Find the following:
(a) [3 marks] the equation(s) of the vertical asymptote(s) to the graph of $y=\frac{x^{2}-1}{x^{2}-4 x+3}$.

Solution: factor and divide. For $x \neq 1, x \neq 3$,

$$
y=\frac{x^{2}-1}{x^{2}-4 x+3}=\frac{(x-1)(x+1)}{(x-1)(x-3)}=\frac{x+1}{x-3}
$$

Then the only vertical asymptote to the graph of $y$ is the line $x=3$.
(b) [3 marks] $\lim _{x \rightarrow 0} \frac{\sin ^{-1}(2 x)}{\tan x}$

Solution: the limit is in the 0/0 form. Use L'Hopital's Rule:

$$
\lim _{x \rightarrow 0} \frac{\sin ^{-1}(2 x)}{\tan x}=\lim _{x \rightarrow 0} \frac{\frac{2}{\sqrt{1-(2 x)^{2}}}}{\sec ^{2} x}=2
$$

(c) [4 marks] the maximum and minimum values of $f(x)=x^{3}-3 x$ on the interval $[0,2]$.

Solution: The extreme values will occur at the endpoints or the the critical point(s).

## Endpoints:

$$
f(0)=0 \text { and } f(2)=2 .
$$

## Critical Points:

$$
f^{\prime}(x)=3 x^{2}-3=0 \Rightarrow x= \pm 1,
$$


and $f(1)=-2$.
So the maximum value of $f$ on $[0,2]$ is $M=2$ and the minimum value of $f$ on $[0,2]$ is $m=-2$.
2. [avg: 6.3/10] Let $R$ be the region bounded by the curves $y=3 x$ and $y=x^{2}$ for $0 \leq x \leq 3$. Set up (but do not calculate) integrals with respect to $x$ that give the following volumes:
(a) [3 marks] the volume of the solid generated by revolving $R$ about the $x$-axis.

Solution: Use the method of washers:

$$
V=\int_{0}^{3} \pi\left((3 x)^{2}-\left(x^{2}\right)^{2}\right) d x
$$


(b) [3 marks] the volume of the solid generated by revolving $R$ about the $y$-axis.

Solution: use the method of shells:

$$
V=\int_{0}^{3} 2 \pi x\left(3 x-x^{2}\right) d x
$$

(c) [2 marks] the volume of the solid generated by revolving $R$ about the line with equation $x=-2$.

Solution: variation on part (b):

$$
V=\int_{0}^{3} 2 \pi(x+2)\left(3 x-x^{2}\right) d x
$$

(d) [2 marks] the volume of the solid generated by revolving $R$ about the line with equation $y=9$.

Solution: variation on part (a):

$$
V=\int_{0}^{3} \pi\left(\left(9-x^{2}\right)^{2}-(9-3 x)^{2}\right) d x
$$

3. [avg: $7.57 / 10$ ] Let $v=10 \sin (2 t)$ be the velocity of a particle at time $t$, for $0 \leq t \leq 3 \pi / 2$. Find:
(a) [4 marks] the average velocity of the particle.

## Solution:

$$
\begin{aligned}
v_{\mathrm{avg}} & =\frac{1}{3 \pi / 2} \int_{0}^{3 \pi / 2} v d t \\
& =\frac{2}{3 \pi} \int_{0}^{3 \pi / 2} 10 \sin (2 t) d t \\
& =\frac{2}{3 \pi}[-5 \cos (2 t)]_{0}^{3 \pi / 2} \\
& =\frac{20}{3 \pi}
\end{aligned}
$$


(b) [6 marks] the average speed of the particle.

Solution: speed is $|v|$ :

$$
\begin{aligned}
|v|_{\text {avg }} & =\frac{1}{3 \pi / 2} \int_{0}^{3 \pi / 2}|v| d t \\
& =3\left(\frac{2}{3 \pi} \int_{0}^{\pi / 2} 10 \sin (2 t) d t\right) \\
& =\frac{2}{\pi}[-5 \cos (2 t)]_{0}^{\pi / 2} \\
& =\frac{20}{\pi}
\end{aligned}
$$


4. [avg: 6.97/10] Find and simplify the following:
(a) $[4$ marks $] F^{\prime}(3)$, if $F(x)=\int_{0}^{\sqrt{x}} \tan ^{-1} t d t$

Solution: use Fundamental Theorem of Calculus, Part 1, and good old chain rule:

$$
F^{\prime}(x)=\left(\tan ^{-1} \sqrt{x}\right)\left(\frac{1}{2 \sqrt{x}}\right) ; F^{\prime}(3)=\left(\tan ^{-1} \sqrt{3}\right)\left(\frac{1}{2 \sqrt{3}}\right)=\frac{\pi}{6 \sqrt{3}}
$$

(b) [6 marks] $\int_{0}^{4} x^{3} \sqrt{16-x^{2}} d x$

Solution: let $u=16-x^{2}$; then $d u=-2 x d x$ and

$$
\begin{aligned}
\int_{0}^{4} x^{3} \sqrt{16-x^{2}} d x & =\frac{1}{2} \int_{0}^{4} x^{2} \sqrt{16-x^{2}}(2 x) d x \\
& =\frac{1}{2} \int_{16}^{0}(16-u) \sqrt{u}(-d u) \\
& =\frac{1}{2} \int_{0}^{16}(16-u) \sqrt{u} d u \\
& =\frac{1}{2} \int_{0}^{16}\left(16 \sqrt{u}-u^{3 / 2}\right) d u \\
& =\frac{1}{2}\left[\frac{32}{3} u^{3 / 2}-\frac{2}{5} u^{5 / 2}\right]_{0}^{16} \\
& =\frac{1}{2}\left(\frac{2048}{3}-\frac{2048}{5}\right) \\
& =\frac{2048}{15}
\end{aligned}
$$

5. [avg: 8.5/10] Let $A$ be the area of the region bounded by $y=\ln x, y=2, y=0$ and $x=0$.
(a) [3 marks] Express the value of $A$ in terms of one or more integrals with respect to $x$.


## Solution:

$$
A=\int_{0}^{1} 2 d x+\int_{1}^{e^{2}}(2-\ln x) d x
$$

(b) [3 marks] Express the value of $A$ in terms of one or more integrals with respect to $y$.

Solution: $y=\ln x \Leftrightarrow x=e^{y}$. Then

$$
A=\int_{0}^{2} e^{y} d y
$$

(c) [4 marks] Find $A$.

Solution: integrate with respect to $y$ since at this stage of the course, we don't know how to integrate $\ln x$ with respect to $x$ :

$$
A=\int_{0}^{2} e^{y} d y=\left[e^{y}\right]_{0}^{2}=e^{2}-1
$$

6. [avg: 7.55/10] A water tank is shaped like a sphere with radius 3 m . If the tank is full, how much work is required to pump all the water to an exit pipe 1 m above the top of the tank? (Assume the density of water is $\rho=1000 \mathrm{~kg} / \mathrm{m}^{3}$ and that $g=9.8 \mathrm{~m} / \mathrm{sec}^{2}$.)


Solution: put the origin at the centre of the tank, so that the cross-section of the tank in the $x, y$-plane has equation

$$
x^{2}+y^{2}=9 .
$$

Then the work done is

$$
W=\int_{-3}^{3} \rho g A(y)(4-y) d y
$$

where $A(y)$ is the cross-sectional area of the tank at height $y$.

We have $A(y)=\pi x^{2}=\pi\left(9-y^{2}\right)$, so

$$
\begin{aligned}
& W=\int_{-3}^{3} \rho g A(y)(4-y) d y \\
& =\int_{-3}^{3} \rho g \pi\left(9-y^{2}\right)(4-y) d y \\
& =\int_{-3}^{3} \rho g \pi\left(36-4 y^{2}-9 y+y^{3}\right) d y \\
& =\rho g \pi\left(\int_{-3}^{3}\left(36-4 y^{2}\right) d y+\int_{-3}^{3}\left(y^{3}-9 y\right) d y\right) \\
& (\text { by symmetry })=2 \rho g \pi \int_{0}^{3}\left(36-4 y^{2}\right) d y+0 \\
& =2 \rho g \pi\left[36 y-\frac{4 y^{3}}{3}\right]_{0}^{3} \\
& =144 \rho g \pi
\end{aligned}
$$

So it will require $144 \rho g \pi=1411200 \pi \approx 4,433,416$ Joules to empty the tank.
7. [avg: 6.96/10] Consider the curve $y=x^{3}+\frac{1}{12 x}$, for $1 \leq x \leq 2$.
(a) [5 marks] Find the length of the curve.

## Solution:

$$
\begin{aligned}
\frac{d y}{d x} & =3 x^{2}-\frac{1}{12 x^{2}} \\
\Rightarrow 1+\left(\frac{d y}{d x}\right)^{2} & =1+\left(9 x^{4}-\frac{1}{2}+\frac{1}{144 x^{4}}\right)=9 x^{4}+\frac{1}{2}+\frac{1}{144 x^{4}} \\
\Rightarrow \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} & =\sqrt{\left(3 x^{2}+\frac{1}{12 x^{2}}\right)^{2}}=3 x^{2}+\frac{1}{12 x^{2}}
\end{aligned}
$$

Thus the length of the curve is

$$
L=\int_{1}^{2} \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x=\int_{1}^{2}\left(3 x^{2}+\frac{1}{12 x^{2}}\right) d x=\left[x^{3}-\frac{1}{12 x}\right]_{1}^{2}=\frac{169}{24}
$$

(b) [5 marks] Find the area of the surface generated by revolving the curve about the $y$-axis.

Solution: use your calculation from part (a).

$$
\begin{aligned}
S A & =\int_{1}^{2} 2 \pi x \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x \\
& =\int_{1}^{2} 2 \pi x\left(3 x^{2}+\frac{1}{12 x^{2}}\right) d x \\
& =2 \pi \int_{1}^{2}\left(3 x^{3}+\frac{1}{12 x}\right) d x \\
& =2 \pi\left[\frac{3 x^{4}}{4}+\frac{\ln x}{12}\right]_{1}^{2} \\
& =2 \pi\left(12+\frac{\ln 2}{12}-\frac{3}{4}\right) \\
& =\frac{45 \pi}{2}+\frac{\pi \ln 2}{6}
\end{aligned}
$$

8. [avg: 6.12/10] Let $f(x)=x^{6 / 5}+6 x^{1 / 5}$, for which $f^{\prime}(x)=\frac{6}{5} x^{1 / 5}+\frac{6}{5} x^{-4 / 5}$ and $f^{\prime \prime}(x)=\frac{6}{25} x^{-4 / 5}-\frac{24}{25} x^{-9 / 5}$.
(a) [1 mark] On which interval(s) is $f$ increasing?

## Solution:

$$
f^{\prime}(x)=\frac{6}{5} x^{1 / 5}+\frac{6}{5} x^{-4 / 5}=\frac{6(x+1)}{5 x^{4 / 5}}
$$

So $f^{\prime}(x)>0$ if $x>-1, x \neq 0$. The graph is increasing for $-1<x<0$ or $x>0$.
(b) [1 mark] On which interval(s) is $f$ decreasing?

Solution: the graph is decreasing if $f^{\prime}(x)<0 \Leftrightarrow x<-1$.
(c) [2 marks] On which interval(s) is $f$ concave up?

## Solution:

$$
f^{\prime \prime}(x)=\frac{6}{25} x^{-4 / 5}-\frac{24}{25} x^{-9 / 5}=\frac{6(x-4)}{25 x^{9 / 5}} .
$$

The graph is concave up if $f^{\prime \prime}(x)>0 \Leftrightarrow x<0$ or $x>4$.
(d) [1 mark] On which interval(s) is $f$ concave down?

Solution: the graph is concave down if $f^{\prime \prime}(x)<0 \Leftrightarrow 0<x<4$.
(e) [5 marks] Sketch the graph of $f$ indicating critical points and inflection points, if any.

## Solution:

The graph has an absolute minimum at


$$
(x, y)=(-1,-5)
$$

it has two inflection points, at

$$
(x, y)=(0,0) \text { and }\left(4,10 \cdot 4^{1 / 5}\right)
$$

there is a vertical tangent at $x=0$ since

$$
\lim _{x \rightarrow 0} f^{\prime}(x)=\infty
$$

9. [avg: 5.48/10] Consider the problem:

Two triangular pens (areas for animals) are built against a barn. Two hundred meters of fencing are to be used for the three sides and the diagonal dividing fence. See figure to the right. The total area of the two pens is to be maximized.

(a) [3 marks] Set this problem up in terms of the dimensions of the rectangle in the figure. That is: which function is to be maximized, subject to which constraint(s)?
Solution: maximize $A=2\left(\frac{x y}{2}\right)=x y$ such that $2 y+x+\sqrt{x^{2}+y^{2}}=200$.
(b) [3 marks] Now set this problem up in terms of an acute angle in one of the triangles, and the length of the diagonal dividing fence.

Solution: use $x=z \cos \theta, y=z \sin \theta$. Now the problem is

$$
\text { maximize } A=z^{2} \sin \theta \cos \theta \text { such that } 2 z \sin \theta+z \cos \theta+z=200 .
$$

(c) [4 marks] Express the total area of the pens in terms of one variable. What equation would you have to solve to find the critical point(s) of your total area function? (You do not actually have to find the critical point(s).)
Solution: from part (b), $z=\frac{200}{2 \sin \theta+\cos \theta+1}$. So as a function of $\theta$,

$$
A=\frac{200^{2} \sin \theta \cos \theta}{(2 \sin \theta+\cos \theta+1)^{2}},
$$

for $0<\theta<\pi / 2$. Then

$$
\begin{aligned}
\frac{d A}{d \theta} & =200^{2} \frac{\left(\cos ^{2} \theta-\sin ^{2} \theta\right)(2 \sin \theta+\cos \theta+1)^{2}-2 \sin \theta \cos \theta(2 \sin \theta+\cos \theta+1)(2 \cos \theta-\sin \theta)}{(2 \sin \theta+\cos \theta+1)^{4}} \\
& =200^{2} \frac{\left(\cos ^{2} \theta-\sin ^{2} \theta\right)(2 \sin \theta+\cos \theta+1)-2 \sin \theta \cos \theta(2 \cos \theta-\sin \theta)}{(2 \sin \theta+\cos \theta+1)^{3}} .
\end{aligned}
$$

Expanding and simplifying the numerator gives:

$$
\frac{d A}{d \theta}=0 \Rightarrow \cos ^{3} \theta+\cos ^{2} \theta+\sin ^{2} \theta \cos \theta-2 \sin ^{3} \theta-\sin ^{2} \theta-2 \cos ^{2} \theta \sin \theta=0
$$

Note: $2 \sin \theta+\cos \theta+1 \neq 0$ for $0<\theta<\pi / 2$, so there is only one type of critical point.

Rest of Solution to Question 9, for those interested: using $\sin ^{2} \theta=1-\cos ^{2} \theta$ the equation at the end of the previous page can be simplified to

$$
2 \cos ^{2} \theta+\cos \theta-2 \sin \theta-1=0
$$

although the 'obvious' solution $\theta=\pi$ is irrelevant. The acute solution can be approximated using Newton's method to find that $\theta \approx 0.6142904078$, and consequently $z \approx 67.34, x \approx 55.03, y \approx 38.81$ and $A \approx 2135.95$.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

