University of Toronto

Faculty of Applied Science and Engineering

Final Examination, December 2014

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

Solutions for **MAT186H1F - Calculus I**

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Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

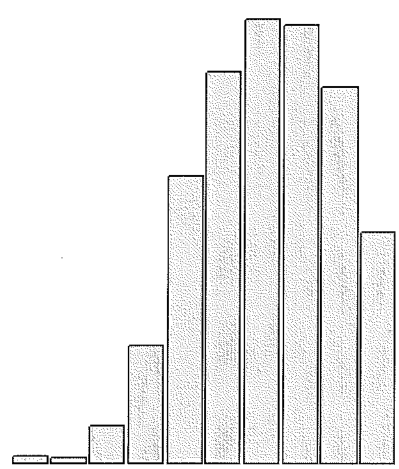
General Comments:

1. Question 1 caused the most problem. There were three common but totally wrong "solutions". One was that $F(x)$ was just the same as $f(x)$; one was that the graph of $F(x)$ was a piece-wise linear function; and the weirdest of all was that the graph of $F(x)$ was just the graph of $f(x)$ rotated counterclockwise by 90°! Question 1 was the only question with a failing average.

2. The range on each question was 0 to 10.

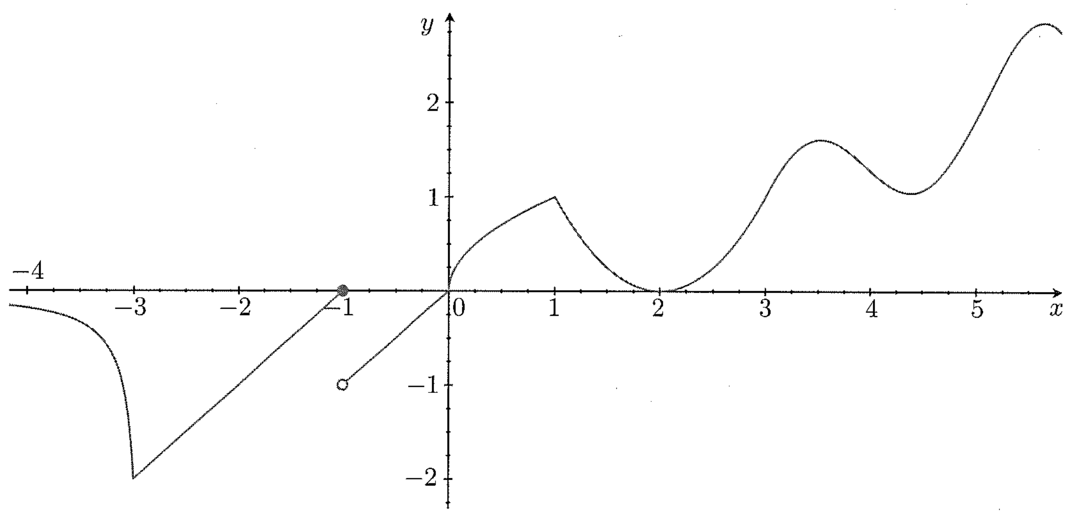
Breakdown of Results: 889 students wrote this exam. The marks ranged from 3.75% to 100%, and the average was 65.8%. There were four perfect papers. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	9.9%
A	26.0%	80-89%	16.1%
B	18,8%	70-79%	18.8%
C	19.0%	60-69%	19.0%
D	16.8%	50-59%	16.8%
F	19.5%	40-49%	12.3%
		30-39%	5.1%
		20-29%	1.6%
		10-19%	0.2%
		0-9%	0.3%



PART I : No explanation is necessary.

1. [avg: 4.4/10] The graph of the function f is given below. Let $F(x) = \int_0^x f(t)\,dt$, for $-4 \le x \le 5$.



Plot the graph of F. In the horizontal direction use the same scale as the above graph; in the vertical direction it is the shape of your graph that is important, not the actual y-values.

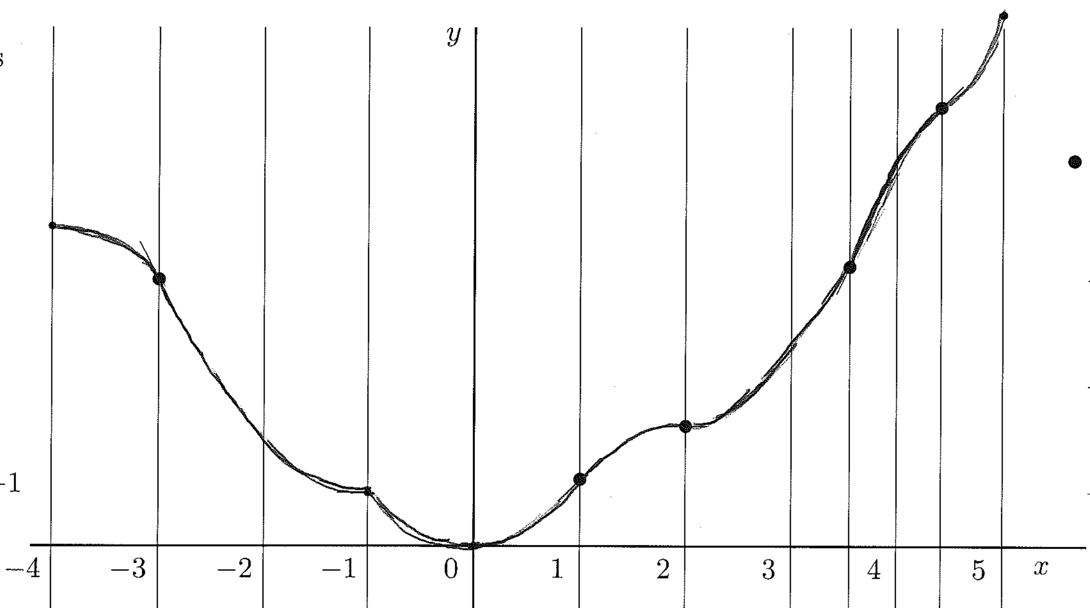
Solution: F is decreasing if $x < 0$; F is increasing if $x > 0$; $F(0) = 0$ is an absolute minimum value.

Other features listed in the three margins:

$F'(-3) = -2$

$\lim_{x \to -1^-} F'(x) = 0$

$\lim_{x \to -1^+} F'(x) = -1$



● five inflection points

$F'(2) = 0$

$F'(1) = 1$

$F'(0) = 0$

$F'(x) = f(x)$:	< 0	< 0	< 0	< 0	> 0	> 0	> 0	> 0		> 0
$F''(x) = f'(x)$:	< 0	$= 1$	$= 1$	$= 1$	> 0	< 0	> 0	> 0	< 0	> 0

Continued...

PART II : Present **complete** solutions to the following questions in the space provided.

2. [avg: 6.3/10] Find the following limits:

(a) [3 marks] $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)\frac{\pi}{n}$

Solution: use the definition of the definite integral, with $\Delta x = \frac{\pi}{n}$, $x_i = \frac{i\pi}{n}$ and $[a, b] = [0, \pi]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)\frac{\pi}{n} = \int_0^{\pi} \sin(x)\, dx = [-\cos(x)]_0^{\pi} = -\cos\pi + \cos 0 = 2.$$

(b) [3 marks] $\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3}$

Solution: the limit is in the 0/0 indeterminate form, so use L'Hopital's Rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3} &= \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1}{3x^2} \\
\text{(L'H again)} &= \lim_{x\to 0} \frac{\frac{-2x}{(1+x^2)^2}}{6x} \\
&= \lim_{x\to 0} \frac{\frac{-1}{(1+x^2)^2}}{3} \\
&= -\frac{1}{3}
\end{aligned}$$

(c) [4 marks] $\lim_{x\to\infty} (x + e^x)^{2/x}$

Solution: the limit is in the ∞^0 indeterminate form so take the natural log of the limit, and then use L'Hopital's Rule:

$$\begin{aligned}
L = \lim_{x\to\infty} (x + e^x)^{2/x} \Rightarrow \ln L &= \lim_{x\to\infty} \frac{2}{x} \ln(x + e^x) \\
\text{(use L'H)} &= 2 \lim_{x\to\infty} \frac{1 + e^x}{x + e^x} \\
\text{(use L'H)} &= 2 \lim_{x\to\infty} \frac{e^x}{1 + e^x} \\
\text{(use L'H)} &= 2 \lim_{x\to\infty} \frac{e^x}{e^x} \\
&= 2 \\
\Rightarrow L &= e^2
\end{aligned}$$

3. [avg: 6.2/10] Find the following:

(a) [5 marks] $\displaystyle\int_{-3}^{4} \frac{x}{(4+x)^{1/3}}\,dx$

Solution: let $u = 4 + x$; then $du = dx$ and $x = u - 4$, and

$$\begin{aligned}
\int_{-3}^{4} \frac{x}{(4+x)^{1/3}}\,dx &= \int_{1}^{8} \frac{u-4}{u^{1/3}}\,du \\
&= \int_{1}^{8} \left(u^{2/3} - \frac{4}{u^{1/3}}\right) du \\
&= \left[\frac{3}{5}u^{5/3} - \frac{3}{2}(4u^{2/3})\right]_{1}^{8} \\
&= \frac{96}{5} - 24 - \frac{3}{5} + 6 = \frac{3}{5}
\end{aligned}$$

(b) [5 marks] $F'(2)$ if $F(x) = x\displaystyle\int_{8}^{x^3} \sqrt{36+t^2}\,dt$

Solution: use the product rule, the fundamental theorem of calculus, and the chain rule:

$$F'(x) = 1\int_{8}^{x^3} \sqrt{36+t^2}\,dt + x\sqrt{36+(x^3)^2}\,(3x^2).$$

So

$$F'(2) = 1\int_{8}^{8} \sqrt{36+t^2}\,dt + 2\sqrt{36+(8)^2}\,(12) = 0 + 24\sqrt{100} = 240.$$

4. [avg: 7.7/10] Let A be the area of the region between $y = 1 + \ln x$ and $y = 0$ for $1 \le x \le e^2$.

(a) [6 marks] Express the value of A in terms of one or more integrals with respect to x **and** in terms of one or more integrals with respect to y. (Draw a diagram!)

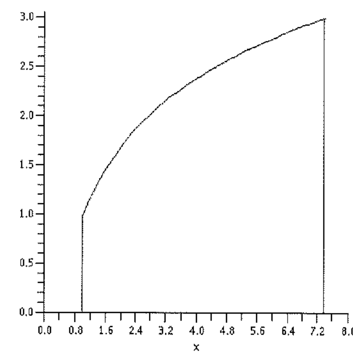
Solution:

$$A = \int_1^{e^2} (1 + \ln x)\, dx;$$

$$A = \int_0^1 (e^2 - 1)\, dy + \int_1^3 (e^2 - e^{y-1})\, dy;$$

where we have used the fact that

$$y = 1 + \ln x \Leftrightarrow x = e^{y-1}.$$



(b) [4 marks] Find A.

Solution: since at this point of the game we can't integrate $\ln x$, we calculate the area using integrals with respect to y:

$$\begin{aligned} A &= \int_0^1 (e^2 - 1)\, dy + \int_1^3 (e^2 - e^{y-1})\, dy \\ &= e^2 - 1 + \left[e^2 y - e^{y-1}\right]_1^3 \\ &= e^2 - 1 + 3e^2 - e^2 - e^2 + 1 \\ &= 2e^2 \end{aligned}$$

Aside/Alternate Calculation: if you know the antiderivative of $\ln x$, then

$$A = \int_1^{e^2} (1 + \ln x)\, dx = [x + x\ln x - x]_1^{e^2} = e^2 \ln(e^2) - 0 = 2e^2,$$

as before.

5. [avg: 8.7/10] Let $v = t^2 - 5t + 6$ be the velocity of a particle at time t, for $0 \le t \le 3$. Find:

(a) [4 marks] the average velocity of the particle.

Solution:

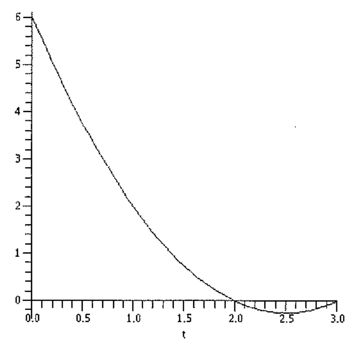
$$v_{avg} = \frac{1}{3}\int_0^3 v\,dt = \frac{1}{3}\int_0^3 (t^2 - 5t + 6)\,dt = \frac{1}{3}\left[\frac{t^3}{3} - \frac{5}{2}t^2 + 6t\right]_0^3 = \frac{1}{3}\left(\frac{27}{3} - \frac{45}{2} + 18\right) = \frac{3}{2}.$$

(b) [6 marks] the average speed of the particle.

Solution: we need to calculate the average value of $|v|$ on $[0,3]$. This will require two separate calculations since

$$v = t^2 - 5t + 6 = (t-2)(t-3),$$

and so $v \ge 0$ on the interval $[0,2]$, but $v \le 0$ on the interval $[2,3]$.



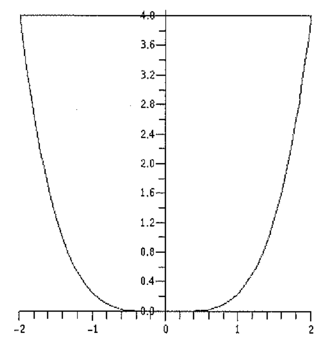
$$\begin{aligned}
\text{speed}_{avg} &= \frac{1}{3}\int_0^3 |v|\,dt \\
&= \frac{1}{3}\left(\int_0^2 v\,dt + \int_2^3 (-v)\,dt\right) \\
&= \frac{1}{3}\left(\int_0^2 v\,dt - \int_2^3 v\,dt\right) \\
&= \frac{1}{3}\left[\frac{t^3}{3} - \frac{5}{2}t^2 + 6t\right]_0^2 - \frac{1}{3}\left[\frac{t^3}{3} - \frac{5}{2}t^2 + 6t\right]_2^3 \\
&= \frac{1}{3}\left(\frac{8}{3} - \frac{5}{2}(2^2) + 12\right) - \frac{1}{3}\left(\frac{27}{3} - \frac{45}{2} + 18 - \left(\frac{8}{3} - \frac{5}{2}(2^2) + 12\right)\right) \\
&= \frac{14}{9} + \frac{1}{18} \\
&= \frac{29}{18}
\end{aligned}$$

Continued...

6. [avg: 6.1/10] A storage tank's shape is formed by rotating the curve $y = \frac{x^4}{4}$, for $0 \le x \le 2$, around the y-axis. (Assume distances are measured in meters.)

(a) [5 marks] What is the volume of the tank?

Solution: use the method of discs with respect to the y-axis, and the fact that $x^2 = 2\sqrt{y}$. The volume of the tank, in cubic meters, is



$$\begin{aligned} V &= \int_0^4 \pi x^2\, dy \\ &= \int_0^4 2\pi\sqrt{y}\, dy \\ &= 2\pi \left[\frac{2}{3}y^{3/2}\right]_0^4 \\ &= \frac{32}{3}\pi \\ &\approx 33.5 \end{aligned}$$

Alternate Solution: use the method of shells and integrate with respect to x;

$$V = \int_0^2 2\pi x \left(4 - \frac{x^4}{4}\right) dx = 2\pi \int_0^2 \left(4x - \frac{x^5}{4}\right) dx = 2\pi \left[2x^2 - \frac{x^6}{24}\right]_0^2 = \frac{32}{3}\pi.$$

(b) [5 marks] Suppose the tank is filled with water of density $\rho = 1000$ kg/m^3. How much work is needed to pump all the water out of the tank to an outflow pipe 1 meter above the top of the tank? Use $g = 9.8$ m/sec^2.

Solution: the cross-sectional area of the tank at height y is $A(y) = \pi x^2 = 2\pi\sqrt{y}$, as in part (a). Then the work needed to pump all the water out of the tank to an outflow pipe 1 meter above the top of the tank, in Joules, is

$$\begin{aligned} W &= \int_0^4 \rho\, g\, A(y)\,(5-y)\, dy \\ &= 2\rho\pi g \int_0^4 (5\sqrt{y} - y^{3/2})\, dy \\ &= 2\rho\pi g \left[\frac{10}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 \\ &= \frac{416}{15}\rho\pi g \\ &\approx 853,843 \end{aligned}$$

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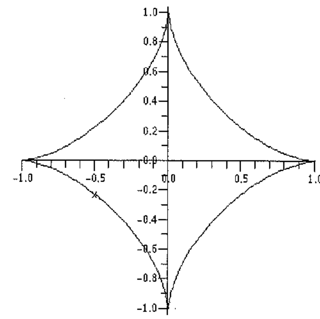
7. [avg: 6.2/10] The graph of the astroid with equation $x^{2/3} + y^{2/3} = 1$ is shown to the right.



(a) [5 marks] Find the total length of the astroid. NB: if your integrand is not defined at $x = 0$ don't worry; just ignore this minor technicality and proceed to integrate as usual.

Solution: use symmetry; total length is four times length of the curve in the first quadrant. Then

$$L = 4\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 4\int_0^1 \frac{1}{x^{1/3}}\, dx = 4\left[\frac{3}{2}x^{2/3}\right]_0^1 = 6,$$

where you can calculate $\sqrt{1+(y')^2}$ explicitly or implicitly, as follows:

$$y = (1-x^{2/3})^{3/2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}\sqrt{1-x^{2/3}}\left(-\frac{2}{3}\frac{1}{x^{1/3}}\right) = -\frac{\sqrt{1-x^{2/3}}}{x^{1/3}}$$

$$\Rightarrow \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{x^{2/3}+1-x^{2/3}}{x^{2/3}}} = \frac{1}{x^{1/3}}$$

OR:

$$x^{2/3} + y^{2/3} = 1 \quad \Rightarrow \quad \frac{2}{3}\frac{1}{x^{1/3}} + \frac{2}{3}\frac{1}{y^{1/3}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\Rightarrow \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}} = \frac{1}{x^{1/3}}$$

(b) [5 marks] Find the area of the surface generated by revolving the astroid about the x-axis.

Solution: double the surface area of the right half.

$$\begin{aligned} SA &= 2\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= 4\pi\int_0^1 (1-x^{2/3})^{3/2}\frac{1}{x^{1/3}}\, dx \\ (\text{let } u = 1-x^{2/3}) \quad &= 4\pi\left(-\frac{3}{2}\right)\int_1^0 u^{3/2}\, du \\ &= 6\pi\int_0^1 u^{3/2}\, du = 6\pi\left[\frac{2}{5}u^{5/2}\right]_0^1 = \frac{12}{5}\pi \end{aligned}$$

Note: its actually easier to use symmetry and revolve around the y-axis instead:

$$SA = 2\int_0^1 2\pi x\left(\frac{1}{x^{1/3}}\right) dx = 4\pi\int_0^1 x^{2/3}\, dx = 4\pi\left[\frac{3}{5}x^{5/3}\right]_0^1 = \frac{12}{5}\pi.$$

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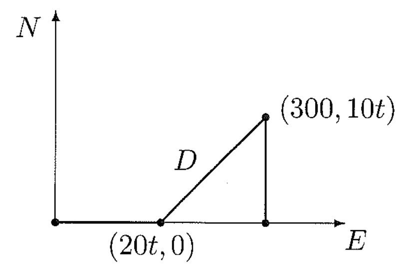
8. [avg: 7.0/10] At 1 PM a military jet is flying due east at 20 km/min. At that instant, at the same altitude and 300 km directly ahead of the military jet, a commercial air liner is flying due north at 10 km/min. When are the two planes closest to each other? What is the minimum distance between them?

Solution: Let $t = 0$ correspond to 1 PM; let due east be in the direction of the positive x-axis; let due north be in the direction of the positive y-axis; and let the origin be the location of the military jet at 1 PM. Then with time t measured in minutes, the position of the military jet at time t is $(20t, 0)$ and the position of the commercial air liner is $(300, 10t)$. Let D be the distance between the two planes at time t. Then



$$D^2 = (300 - 20t)^2 + (10t)^2,$$

and, differentiating implicitly,

$$2D\frac{dD}{dt} = 2(300 - 20t)(-20) + 2(10t)(10) = 1,000\,t - 12,000.$$

At the critical point,

$$\frac{dD}{dt} = 0 \Rightarrow t = 12.$$

There is a minimum value of D at $t = 12$ since

$$t < 12 \Rightarrow \frac{dD}{dt} < 0 \text{ and } t > 12 \Rightarrow \frac{dD}{dt} > 0.$$

Thus:

- the distance between the two planes is minimized at 1:12 PM,
- and the minimum distance, in km, between them at that time is

$$D = \sqrt{(300 - 240)^2 + (120)^2} = 60\sqrt{5} \approx 134.2$$

This page is for rough work; it will not be marked.

The end.