University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2013**
Duration: 2 and 1/2 hours
First Year - CHE, CIV, IND, LME, MEC, MMS

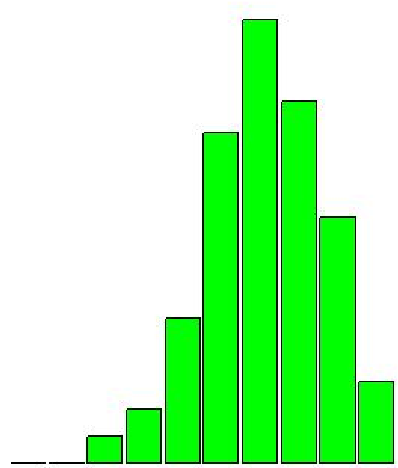
MAT186H1F - CALCULUS I
Exam Type: A

General Comments:

1. In Question 1(c) many students thought $\int \sin(\pi t)\,dt = -\pi\ \cos(\pi t)+C$ or $-\cos(\pi t)+C$.
2. In Question 2(b) your calculator should be in radian mode. If your calculator is in degree mode otherwise correct calculations will give you $x = 0.999847741\ldots$ which is *not* the correct solution to $\cos x = x$.
3. The graph in Question 6 is *not* a circle, although this was a very popular misconception. Question 6 can be greatly simplified using symmetry. Computationally it is easier to calculate the derivatives implicitly, although explicit calculations are possible if you keep in mind which quadrant x and y are in.
4. Not surprisingly the optimization problem, Question 7, caused the most difficulty, as many students could not set it up correctly.
5. It was astonishing how many students threw marks away because of bad notation or by not using equal signs. Many students would profit greatly if they cleaned up their presentation!

Breakdown of Results: 434 students wrote the exam. The marks ranged from 20% to 100%, and the average was 65.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.9%
A	19.4%	80-89%	14.5%
B	21.4%	70-79%	21.4%
C	26.3%	60-69%	26.3%
D	19.6%	50-59%	19.6%
F	13.3%	40-49%	8.5%
		30-39%	3.2%
		20-29%	1.6%
		10-19%	0.0%
		0-9%	0.0%



1. [15 marks] Find the following:

(a) [5 marks] $\displaystyle\int \left(e^x + \frac{1}{1+x^2} + \frac{1}{x} + \sinh x\right) dx$

Solution:

$$\int \left(e^x + \frac{1}{1+x^2} + \frac{1}{x} + \sinh x\right) dx =$$

$$\int e^x\,dx + \int \frac{1}{1+x^2}\,dx + \int \frac{1}{x}\,dx + \int \sinh x\,dx = e^x + \tan^{-1} x + \ln|x| + \cosh x + C$$

(b) [5 marks] $\displaystyle\int_0^{\pi/4} \tan^2 x \sec^2 x\,dx.$

Solution: let $u = \tan x$. Then $du = \sec^2 x\,dx$ and

$$\int_0^{\pi/4} \tan^2 x \sec^2 x\,dx = \int_0^1 u^2\,du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}.$$

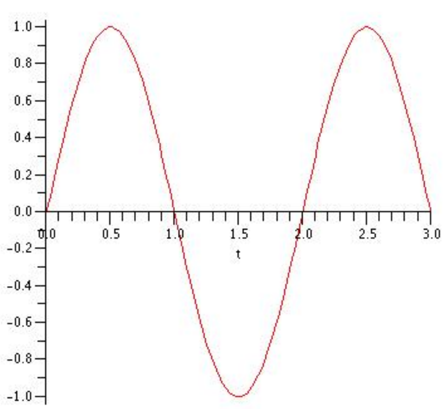
Alternate Solution: more complicated is: $u = \tan^2 x \Rightarrow du = 2\tan x \sec^2 x\,dx$, whence

$$\int_0^{\pi/4} \tan^2 x \sec^2 x\,dx = \frac{1}{2}\int_0^1 \sqrt{u}\,du = \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_0^1 = \frac{1}{3}.$$

(c) [5 marks] the average speed of a particle over the time period $t = 0$ to $t = 3$ if the velocity of the particle at time t is given by $v = \sin(\pi t)$.

Solution: the graph of v is shown below, it includes 1.5 cycles of the graph. Observe that $v > 0$ for $0 < t < 1, 2 < t < 3$ and that $v < 0$ for $1 < t < 2$. In particular,

$$\int_0^1 v\,dt = \int_1^2 -v\,dt = \int_2^3 v\,dt.$$



Thus the average speed is given by

$$\frac{1}{3}\int_0^3 |v|\,dt = \frac{1}{3}\left(3\int_0^1 \sin(\pi t)\,dt\right)$$

$$= \int_0^1 \sin(\pi t)\,dt = \left[-\frac{\cos(\pi t)}{\pi}\right]_0^1$$

$$= \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}.$$

2. [10 marks] Find the following:

(a) [4 marks] $F'(1)$ if $F(x) = \int_0^{x^4} \sqrt{t^2 + 4t + 11}\, dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = 4x^3 \sqrt{(x^4)^2 + 4x^4 + 11} = 4x^3 \sqrt{x^8 + 4x^4 + 11}.$$

So

$$F'(1) = 4\sqrt{1 + 4 + 11} = 4\sqrt{16} = 16.$$

(b) [6 marks] an approximation of the solution to the equation $\cos x = x$, correct to 3 decimal places. Note: x is in radians!

Solution: let $f(x) = x - \cos x$ and use Newton's method to approximate the solution to the equation $f(x) = 0$. Observe that

$$f(0) = -1 < 0 \text{ and } f(\pi/2) = \frac{\pi}{2} > 0.$$

So the solution is in the interval $[0, \pi/2]$. We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n}{1 + \sin x_n}.$$

Then

$$x_0 = 1 \Rightarrow x_1 = 0.7503638679 \cdots \Rightarrow x_2 = 0.7391128909 \ldots$$
$$\Rightarrow x_3 = 0.7390851334 \cdots \Rightarrow x_4 = 0.7390851332 \ldots$$

The solution is $x = 0.739$, correct to three decimal places.

3. [13 marks] Let $f(x) = \dfrac{2x^2 - x - 4}{x-2}$, for which $f'(x) = \dfrac{2(x-1)(x-3)}{(x-2)^2}$, $f''(x) = \dfrac{4}{(x-2)^3}$.

(a) [2 marks] Find the interval(s) on which f is increasing.

Solution: $f'(x) > 0 \Leftrightarrow (x-1)(x-3) > 0$ and $x \neq 2 \Leftrightarrow x < 1$ or $x > 3$.

(b) [2 marks] Find the interval(s) on which f is decreasing.

Solution: $f'(x) < 0 \Leftrightarrow (x-1)(x-3) < 0$ and $x \neq 2 \Leftrightarrow 1 < x < 2$ or $2 < x < 3$.

(c) [1 mark] Find the interval(s) on which f is concave up.

Solution: $f''(x) > 0 \Leftrightarrow (x-2)^3 > 0 \Leftrightarrow x > 2$.

(d) [1 mark] Find the interval(s) on which f is concave down.

Solution: $f''(x) < 0 \Leftrightarrow (x-2)^3 < 0 \Leftrightarrow x < 2$.

(e) [2 marks] Find the equations of all asymptotes to the graph of f.

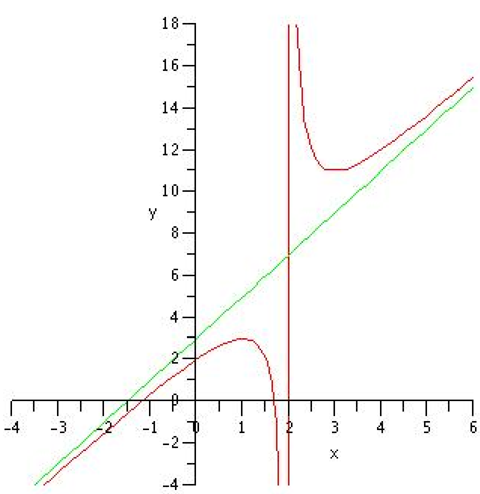
Solution: long division gives

$$\frac{2x^2 - x - 4}{x-2} = 2x + 3 + \frac{2}{x-2},$$

so $y = 2x + 3$ is a slant (or oblique) asymptote; and $x = 2$ is a vertical tangent because

$$\lim_{x \to 2^-} \frac{2x^2 - x - 4}{x-2} = -\infty, \quad \lim_{x \to 2^+} \frac{2x^2 - x - 4}{x-2} = \infty.$$

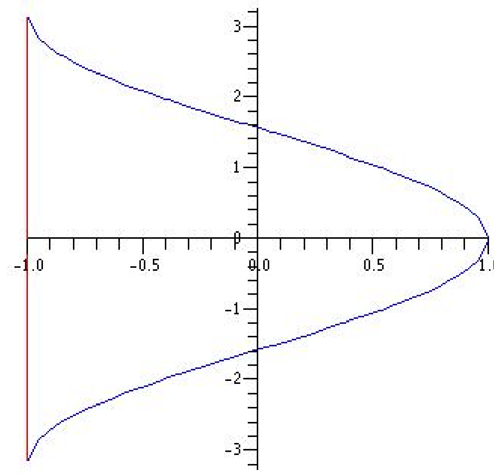
(f) [5 marks] Sketch the graph of f labeling all critical points, inflection points and asymptotes, if any. The **graph** is to the right. There is a min at $(3, 11)$ and a max at $(1, 3)$. There are no inflection points.



4. [12 marks] Let A be the area of the region in the xy-plane bounded by the curves $y = \cos^{-1} x$ and $y = -\cos^{-1} x$ on the interval $-1 \le x \le 1$. Note: $\cos^{-1} x = \arccos x$.

(a) [8 marks] Write down two integrals, one with respect to x and one with respect to y, that both give the value of A.

Solution: the region A is indicated in the graph below.



With respect to x:

$$A = \int_{-1}^{1} \left(2\cos^{-1} x\right)\, dx.$$

With respect to y:

$$A = \int_{-\pi}^{\pi} (\cos y + 1)\, dy,$$

since $y = \cos^{-1} x \Rightarrow x = \cos y$ and
$y = -\cos^{-1} x \Rightarrow x = \cos(-y) = \cos y$.

(b) [4 marks] Find the value of A.

Solution: integrate with respect to y, since at this stage of the game, we don't know how to integrate $\cos^{-1} x$ with respect to x.

$$A = \int_{-\pi}^{\pi} (\cos y + 1)\, dy = [\sin y + y]_{-\pi}^{\pi} = 0 + \pi - 0 - (-\pi) = 2\pi.$$

Alternate Calculation: for those interested, or for those who somehow already know integration by parts, it can be shown that

$$\int \cos^{-1} x \, dx = x\cos^{-1} x - \sqrt{1 - x^2} + C,$$

so

$$\int_{-1}^{1} 2\cos^{-1} x \, dx = 2\left[x\cos^{-1} x - \sqrt{1 - x^2}\right]_{-1}^{1} = 2(0 - 0 - (-\pi) + 0) = 2\pi.$$

5. [15 marks] Find the following limits, if they exist:

(a) [4 marks] $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}(3x)}{e^x-1}$

Solution: this limit is in the 0/0 form. Use L'Hopital's Rule:

$$\lim_{x\to 0}\frac{\sin^{-1}(3x)}{e^x-1}=\lim_{x\to 0}\frac{3/\sqrt{1-9x^2}}{e^x}=\frac{3}{1}=3.$$

(b) [5 marks] $\displaystyle\lim_{x\to+\infty} x\left(\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}-1\right)$

Solution: this limit is in the $\infty\cdot 0$ form. Rewrite it as 0/0 and use L'Hopital's Rule:

$$\lim_{x\to+\infty} x\left(\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}-1\right)=\lim_{x\to+\infty}\frac{\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}-1}{1/x}$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{2\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}}\left(-\frac{3}{x^2}-\frac{10}{x^3}\right)}{-1/x^2}=\lim_{x\to+\infty}\frac{3+10/x}{2\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}}=\frac{3}{2}.$$

Alternate Solution without L'Hopital's Rule: rationalize the numerator.

$$\lim_{x\to+\infty} x\left(\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}-1\right)=\lim_{x\to+\infty}\frac{x(1+\frac{3}{x}+\frac{5}{x^2}-1)}{\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}+1}=\lim_{x\to+\infty}\frac{3+\frac{5}{x}}{\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}+1}=\frac{3}{2}$$

(c) [6 marks] $\displaystyle\lim_{x\to 1^-}(2-x)^{\tan(\pi x/2)}$

Solution: this limit is in the 1^∞ form, so let the limit be L and calculate $\ln L$.

$$L=\lim_{x\to 1^-}(2-x)^{\tan(\pi x/2)}$$

$$\begin{aligned}
\Rightarrow \ln L &= \lim_{x\to 1^-}\ln(2-x)^{\tan(\pi x/2)}=\lim_{x\to 1^-}\tan(\pi x/2)\ln(2-x)\\
&= \lim_{x\to 1^-}\frac{\sin(\pi x/2)\ln(2-x)}{\cos(\pi x/2)}=\lim_{x\to 1^-}\sin(\pi x/2)\lim_{x\to 1^-}\frac{\ln(2-x)}{\cos(\pi x/2)}\\
\text{(L'H)} &= 1\cdot\lim_{x\to 1^-}\frac{-1/(2-x)}{-\sin(\pi x/2)\cdot\pi/2}=\frac{2}{\pi}\\
\Rightarrow L &= e^{2/\pi}
\end{aligned}$$

6. [10 marks] Sketch the graph of the relation

$$x^{2/3} + y^{2/3} = 1,$$

for $-1 \le x \le 1, -1 \le y \le 1$, indicating all critical points, inflection points, horizontal tangents and vertical tangents, if any.

Solution: differentiate implicitly.

$$\frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

The **critical points** are

$$(x, y) = (\pm 1, 0), \text{ at which } \frac{dy}{dx} = 0, \text{ and } (x, y) = (0, \pm 1), \text{ at which } \frac{dy}{dx} \text{ is undefined.}$$

Now

$$\frac{dy}{dx} > 0 \Leftrightarrow \frac{y}{x} < 0 \text{ and } \frac{dy}{dx} < 0 \Leftrightarrow \frac{y}{x} > 0,$$

so the graph of the relation is **increasing** in the 2nd and 4th quadrants, and **decreasing** in the first and third quadrants. For the second derivative:

$$\left(\frac{dy}{dx}\right)^3 = -\frac{y}{x} \Rightarrow 3\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} = -\frac{\frac{dy}{dx}x - y}{x^2} = -\frac{-\left(\frac{y}{x}\right)^{1/3} x - y}{x^2} = \frac{y^{1/3}(x^{2/3} + y^{2/3})}{x^2} = \frac{y^{1/3}}{x^2},$$

since $x^{2/3} + y^{2/3} = 1$. Thus

$$\frac{d^2y}{dx^2} > 0 \Leftrightarrow y > 0 \text{ and } \frac{d^2y}{dx^2} < 0 \Leftrightarrow y < 0,$$

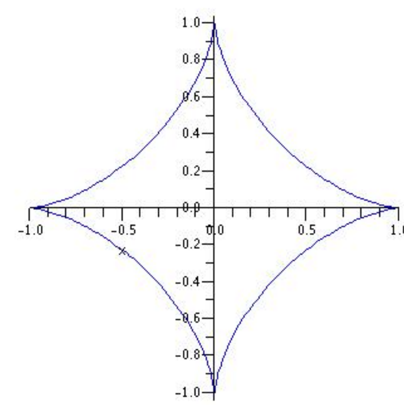
which means the graph of the relation is **concave up** if $y > 0$ and **concave down** if $y < 0$. The **graph** of the relation is shown below:

The x-axis is a **horizontal tangent** to the graph since at both $(\pm 1, 0)$

$$\frac{dy}{dx} = 0.$$

The y-axis is a **vertical tangent** to the graph since

$$\lim_{x\to 0}\left|\frac{dy}{dx}\right| = \lim_{x\to 0}\left|\frac{y}{x}\right|^{1/3} = \infty.$$

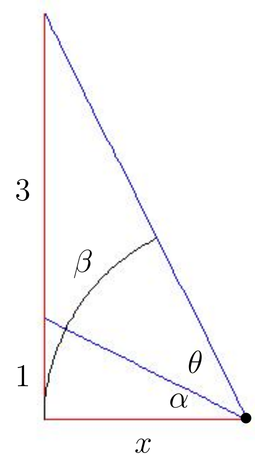


Alternate Solution: observe that the graph is symmetric with respect to both the x-axis and the y-axis, and then simply do the analysis for the graph in the first quadrant, for $0 \le x \le 1, 0 \le y \le 1$, for which

$$y = (1 - x^{2/3})^{3/2}, \quad \frac{dy}{dx} = -\frac{\sqrt{1 - x^{2/3}}}{x^{1/3}}, \quad \frac{d^2y}{dx^2} = \frac{1}{3x^{4/3}\sqrt{1 - x^{2/3}}}.$$

7. [10 marks] The lower edge of a painting, 3 m in height, is 1 m above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

Solution: let the distance from the observer to the wall be x, let the angle from eye level to the bottom of the frame be α, let the angle from eye level to the top of the frame be β, let the angle subtended at the observer's eye by the painting be θ.



Then $\theta = \beta - \alpha$ and

$$\tan\alpha = \frac{1}{x}, \quad \tan\beta = \frac{4}{x}.$$

So

$$\theta = \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right).$$

The problem is to maximize θ for $x > 0$. Find the critical point(s):

$$\frac{d\theta}{dx} = \frac{1}{1+(4/x)^2}\left(-\frac{4}{x^2}\right) - \frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) = -\frac{4}{x^2+16} + \frac{1}{x^2+1}$$

$$\frac{d\theta}{dx} = 0 \Rightarrow \frac{4}{x^2+16} = \frac{1}{x^2+1} = 0 \Rightarrow 4x^2+4 = x^2+16 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

since we are assuming $x > 0$. Confirm a maximum value occurs at $x = 2$:

$$\frac{d^2\theta}{dx^2} = \frac{8x}{(x^2+16)^2} - \frac{2x}{(x^2+1)^2}$$

and

$$\left.\frac{d^2\theta}{dx^2}\right|_{x=2} = -\frac{3}{25} < 0.$$

Conclusion: the observer should stand 2 m from the wall.

Alternate Solution: use the cosine law.

$$9 = 16 + x^2 + 1 + x^2 - 2\sqrt{16+x^2}\sqrt{1+x^2}\cos\theta \Rightarrow \cos\theta = \frac{4+x^2}{\sqrt{16+17x^2+x^4}}$$

After much calculation:

$$-\sin\theta\,\frac{d\theta}{dx} = \frac{9x(x^2-4)}{(16+17x^2+x^4)^{3/2}} \text{ and } \frac{d\theta}{dx} = 0 \Rightarrow x = 2,$$

as before.

8. [15 marks] The parts of this question are unrelated.

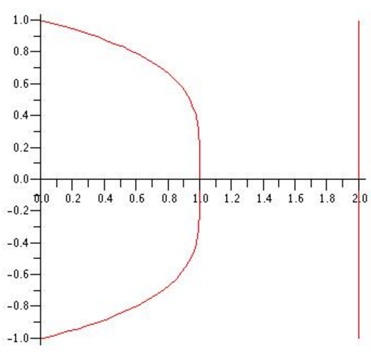
(a) [7 marks] Find the length of the curve $y = \dfrac{1}{8}\left(\dfrac{x^2}{2} - 16\ \ln x\right)$ for $1 \le x \le 4$.

Solution:

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1+(f'(x))^2}\,dx = \int_1^4 \sqrt{1+\left(\frac{1}{8}\left(x-\frac{16}{x}\right)\right)^2}\,dx \\
&= \int_1^4 \sqrt{1+\frac{x^4-32x^2+256}{64\,x^2}}\,dx = \int_1^4 \sqrt{\frac{x^4+32x^2+256}{64\,x^2}}\,dx \\
&= \int_1^4 \frac{x^2+16}{8x}\,dx = \int_1^4 \left(\frac{x}{8}+\frac{2}{x}\right)dx \\
&= \left[\frac{x^2}{16}+2\ln x\right]_1^4 = \frac{16}{16}+2\ln 4-\frac{1}{16}-2\ln 1 \\
&= \frac{15}{16}+2\ln 4 \text{ or } \frac{15}{16}+\ln 16
\end{aligned}
$$

(b) [8 marks] The region bounded by $x = 1 - y^4$ and $x = 0$ is rotated about the line $x = 2$. Find the volume of the resulting solid.

Solution: integrate with respect to x, use the method of shells and $y^4 = 1 - x$:



$$
\begin{aligned}
V &= \int_0^1 2\pi(2-x)\,2\,(1-x)^{1/4}\,dx \\
&= 4\pi\int_0^1 (2-x)(1-x)^{1/4}\,dx \\
\text{(let } u = 1-x) \quad &= 4\pi\int_1^0 (u+1)u^{1/4}\,(-du) \\
&= 4\pi\int_0^1 \left(u^{5/4}+u^{1/4}\right)du \\
&= 4\pi\left[\frac{4}{9}u^{9/4}+\frac{4}{5}u^{5/4}\right]_0^1 \\
&= \frac{224\pi}{45}
\end{aligned}
$$

Alternatively: integrate with respect to y and use the method of washers:

$$
V = \int_{-1}^1 \pi\left(2^2-(2-x)^2\right)dy = \pi\int_{-1}^1 (4x-x^2)\,dy = \pi\int_{-1}^1 (4-4y^4-1+2y^4-y^8)\,dy
$$

$$
= \pi\int_{-1}^1 (3-2y^4-y^8)\,dy = \pi\left[3y-\frac{2y^5}{5}-\frac{y^9}{9}\right]_{-1}^1 = \frac{224\pi}{45}
$$