University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2012**
Duration: 2 and 1/2 hours
First Year - CHE, CIV, IND, LME, MEC, MMS

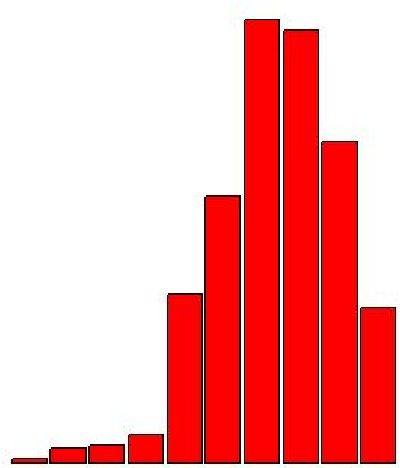
MAT186H1F - CALCULUS I
Exam Type: A

General Comments:

1. On each question, the results ranged from zero to perfect.
2. Only Question 8 did not have a passing average.
3. 86% of this exam was completely routine. Only questions 2(a) and 8 can be considered challenging: the former because it requires some thought to explain clearly why the limit doesn't exist; the latter because a straightforward approach is very messy, assuming you have been able to set the problem up at all. The straightforward approach is:
$$r = R\left(1-\frac{\theta}{2\pi}\right),\ h=\sqrt{R^2-r^2}=\frac{R}{\pi}\sqrt{\theta(4\pi-\theta)}$$
whence
$$V=\frac{R^3}{3}\left(1-\frac{\theta}{2\pi}\right)^2\sqrt{\theta(4\pi-\theta)};\ \frac{dV}{d\theta}=\frac{R^3}{24\pi^2}\frac{(2\pi-\theta)(3\theta^2-12\pi\theta+4\pi^2)}{\sqrt{\theta(4\pi-\theta)}}.$$
Then $dV/d\theta=0 \Rightarrow 3\theta^2-12\pi\theta+4\pi^2=0 \Rightarrow \theta=2\pi(1-\sqrt{2}/\sqrt{3})$, for $0<\theta<2\pi$. So $r=\sqrt{2}R/\sqrt{3}$ and $h=R/\sqrt{3}$. Easier ways to do this problem are presented inside!

Breakdown of Results: 514 students wrote the exam. The marks ranged from 0% to 99%, and the average was 68.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.4%
A	25.7%	80-89%	17.3%
B	23.3%	70-79%	23.3%
C	23.9%	60-69%	23.9%
D	14.4%	50-59%	14.4%
F	12.7%	40-49%	9.1%
		30-39%	1.6%
		20-29%	1.0%
		10-19%	0.8%
		0-9%	0.2%



1. [avg: 8.0/10] Find the following:

   (a) [5 marks] $\displaystyle\int \left(e^x + x^{1/3} + \frac{1}{x}\right) dx$

   Solution:

$$\int \left(e^x + x^{1/3} + \frac{1}{x}\right) dx = \int e^x\,dx + \int x^{1/3}\,dx + \int \frac{dx}{x} = e^x + \frac{3\,x^{4/3}}{4} + \ln|x| + C$$

   (b) [5 marks] $\displaystyle\int_0^{\pi/2} \left(1 + \sin^3 x\right) \cos x\,dx.$

   Solution: let $u = \sin x$. Then $du = \cos x\,dx$ and

$$\int_0^{\pi/2} \left(1 + \sin^3 x\right) \cos x\,dx = \int_0^1 (1 + u^3)\,du = \left[u + \frac{u^4}{4}\right]_0^1 = \frac{5}{4}.$$

2. [avg: 5.8/10] Find the following limits, if they exist:

(a) [2 marks] $\lim_{x\to\infty} e^x \sin x$

Solution: this limit does not exist. Let $f(x) = e^x \sin x$.
If $x = \pi/2 + 2n\pi$, then $f(x) = e^x$, and $n \to \infty \Rightarrow x \to \infty$ and $f(x) \to +\infty$;
if $x = 3\pi/2 + 2n\pi$, then $f(x) = -e^x$, and $n \to \infty \Rightarrow x \to \infty$ and $f(x) \to -\infty$.

(b) [3 marks] $\lim_{x\to-\infty} e^x \sin x$

Solution: this limit is zero, by the Squeezing Theorem:

$$-1 \le \sin x \le 1 \Rightarrow -e^x \le e^x \sin x \le e^x$$

and both

$$\lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to-\infty} -e^x = 0;$$

so

$$\lim_{x\to-\infty} e^x \sin x = 0$$

as well.

(c) [5 marks] $\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x$

Solution: the limit is in the 1^∞ form. Let the limit be L.

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}} \\
\text{(L'H)} &= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{4}{x}}\left(-\frac{4}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{4}{1+\frac{4}{x}} = 4 \\
\Rightarrow L &= e^4
\end{aligned}$$

3. [avg: 8.4/10] Find the following:

(a) [4 marks] $F'(1)$ if $F(x) = \int_0^{x^3} (t^2+8)^{3/2}\,dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = 3x^2\,\left((x^3)^2+8\right)^{3/2}.$$

So

$$F'(1) = 3\,\left(9^{3/2}\right) = 81.$$

(b) [6 marks] an approximation of the solution to the equation $e^x + x^3 = 0$, correct to 2 decimal places.

Solution: let $f(x) = e^x + x^3$ and use Newton's method. Observe that

$$f(0) = 1 > 0 \text{ and } f(-1) = e^{-1} - 1 < 0.$$

So the solution is in the interval $[-1, 0]$. We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} + x_n^3}{e^{x_n} + 3x_n^2}.$$

Then $x_0 = 0 \Rightarrow x_1 = -1 \Rightarrow x_2 = -0.8123090301\ldots$
$\Rightarrow x_3 = -0.7742765490\cdots \Rightarrow x_4 = -0.7728847562\cdots \Rightarrow x_5 = -0.7728829591\ldots$
The solution is $x = -0.77$, correct to two decimal places.

4. [avg: 8.9/10] The velocity of a particle at time t is given by $v = t^3 - 1$. Find

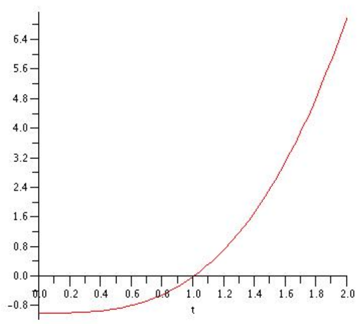
(a) [4 marks] the average velocity of the particle for $0 \le t \le 2$.

Solution: average velocity is

$$\frac{1}{2-0}\int_0^2 v\,dt = \frac{1}{2}\int_0^2 (t^3-1)\,dt = \frac{1}{2}\left[\frac{t^4}{4} - t\right]_0^2 = \frac{1}{2}(4-2) = 1.$$

(b) [6 marks] the average speed of the particle for $0 \le t \le 2$.

Solution: this is a little trickier since v changes sign at $t = 1$. See graph below.



The average speed is

$$\begin{aligned}\frac{1}{2-0}\int_0^2 |v|\,dt &= \frac{1}{2}\int_0^1 (-v)\,dt + \frac{1}{2}\int_1^2 v\,dt \\ &= \frac{1}{2}\left[t - \frac{t^4}{4}\right]_0^1 + \frac{1}{2}\left[\frac{t^4}{4} - t\right]_1^2 \\ &= \frac{1}{2}\left(\frac{3}{4}\right) + \frac{1}{2}\left(2 - \left(-\frac{3}{4}\right)\right) \\ &= \frac{7}{4} \text{ or } 1.75\end{aligned}$$

5. [avg: 8.3/12] Sketch the graph of $y = 6x^{1/3} + 3x^{4/3}$. Label all critical points, inflection points, and vertical tangents. You may assume

$$y' = \frac{2}{x^{2/3}} + 4\,x^{1/3} \text{ and } y'' = -\frac{4}{3\,x^{5/3}} + \frac{4}{3\,x^{2/3}}.$$

Solution: let $y = f(x)$. Then

$$f'(x) = \frac{(2 + 4\,x)}{x^{2/3}};\, f'(x) = 0 \Leftrightarrow x = -\frac{1}{2};\, f'(0) \text{ is undefined;}$$

$$f'(x) > 0 \Leftrightarrow 1 + 2x > 0 \Rightarrow x > -\frac{1}{2}, x \neq 0;\;\; f'(x) < 0 \Leftrightarrow 1 + 2x < 0 \Leftrightarrow x < -\frac{1}{2}.$$

So

$$\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right) = \left(-\frac{1}{2}, \left(-\frac{9}{2^{4/3}}\right)\right)$$

is an **absolute minimum** point. Since $f''(x) = -\frac{4}{3}\frac{(x-1)}{x^{5/3}}$,

$$f''(x) < 0 \Leftrightarrow \frac{x-1}{x^{5/3}} > 0 \Leftrightarrow 0 < x < 1$$

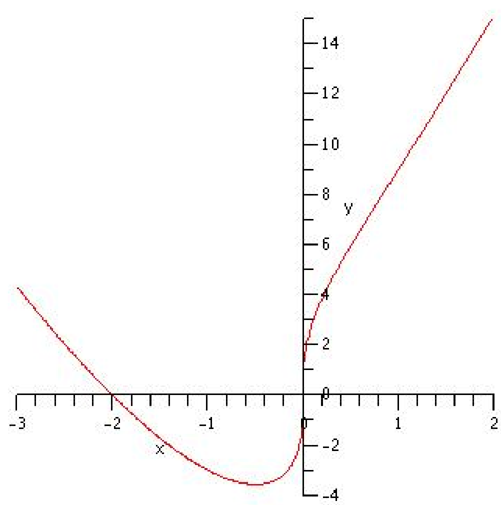
and

$$f''(x) > 0 \Leftrightarrow \frac{x-1}{x^{5/3}} < 0 \Leftrightarrow x < 0 \text{ or } x > 1.$$

Consequently there are **inflection points** at $(0, 0)$ and $(1, 9)$. There is a **vertical tangent** at $x = 0$ since

$$\lim_{x \to 0^+} f'(x) = \infty \text{ and } \lim_{x \to 0^-} f'(x) = \infty.$$

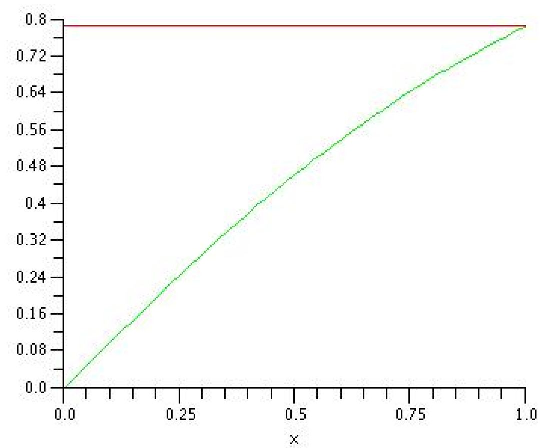
The **graph** of f is shown below:



6. [avg; 6.9/12] Let A be the area of the region in the xy-plane bounded by $x = 0, x = 1, y = \pi/4$ and $y = \tan^{-1} x$.

(a) [8 marks] Write down two integrals, one with respect to x and one with respect to y, that both give the value of A.

Solution: the region A is indicated in the graph below.



With respect to x:

$$A = \int_0^1 \left(\frac{\pi}{4} - \tan^{-1} x\right)\, dx.$$

With respect to y:

$$A = \int_0^{\pi/4} (\tan y)\, dy,$$

since $y = \tan^{-1} x \Rightarrow x = \tan y$.

(b) [4 marks] Find the value of A.

Solution: integrate with respect to y, since at this stage of the game, we don't know how to integrate $\tan^{-1} x$ with respect to x.

$$A = \int_0^{\pi/4} (\tan y)\, dy = \int_0^{\pi/4} \frac{\sin y}{\cos y}\, dy.$$

Now let $u = \cos y$; then $du = -\sin y\, dy$ and

$$A = \int_1^{1/\sqrt{2}} -\frac{du}{u} = \int_{1/\sqrt{2}}^1 \frac{du}{u} = [\ln u]_{1/\sqrt{2}}^1 = \ln 1 - \ln(1/\sqrt{2}) = \ln \sqrt{2} = \frac{\ln 2}{2}.$$

Alternate Calculation: for those interested, or for those who somehow already know integration by parts, it can be shown that

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C,$$

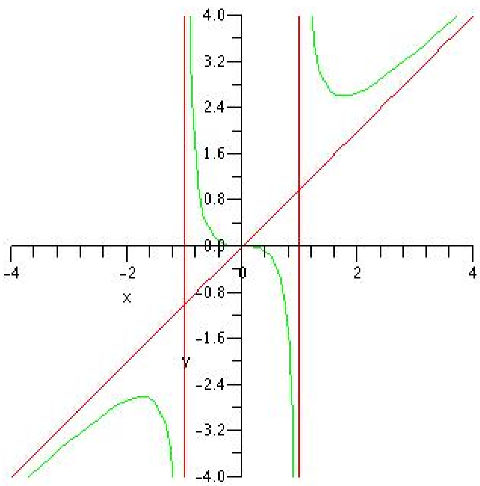
so

$$\int_0^1 \left(\frac{\pi}{4} - \tan^{-1} x\right)\, dx = \left[\frac{\pi x}{4} - x \tan^{-1} x + \frac{1}{2} \ln(1 + x^2)\right]_0^1 = \frac{\ln 2}{2}.$$

7. [avg; 9.6/12] Sketch the graph of $y = \dfrac{x^3}{x^2-1}$. Label all critical points, inflection points and asymptotes. You may assume

$$y' = \frac{x^2(x^2-3)}{(x-1)^2(x+1)^2} \text{ and } y'' = \frac{2x(x^2+3)}{(x-1)^3(x+1)^3}.$$

Solution: let $y = f(x)$. By long division, $f(x) = x - \dfrac{x}{x^2-1}$, so the line with equation $y = x$ is a slant asymptote to the graph of $y = f(x)$. There are vertical asymptotes at



$x = \pm 1$ since

$$\lim_{x\to 1^-} f(x) = -\infty, \lim_{x\to 1^+} f(x) = \infty,$$

$$\lim_{x\to -1^-} f(x) = -\infty, \lim_{x\to -1^+} f(x) = \infty.$$

All these asymptotes are indicated in red on the graph to the left. The graph of $y = f(x)$ is indicated in green. The details for the green graph are given below.

Critical Points: $f'(x) = 0 \Leftrightarrow x = 0, x = \pm\sqrt{3}$. Since

$$f'(x) > 0 \Leftrightarrow x^2 - 3 > 0 \Leftrightarrow x < -\sqrt{3} \text{ or } x > \sqrt{3}$$

and, for $x \neq \pm 1$,

$$f'(x) < 0 \Leftrightarrow x^2 - 3 < 0 \Leftrightarrow -\sqrt{3} < x < -1, -1 < x < 1, \text{ or } 1 < x < \sqrt{3},$$

the graph of $y = f(x)$ is increasing on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ and decreasing on $(-\sqrt{3}, -1) \cup (-1, 1) \cup (1, \sqrt{3})$. So f has a relative max at $(-\sqrt{3}, -3\sqrt{3}/2)$ and a relative min at $(\sqrt{3}, 3\sqrt{3}/2)$.

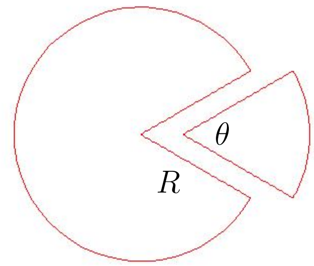
Inflection Points:

$$f''(x) > 0 \Leftrightarrow -1 < x < 0 \text{ or } x > 1; \; f''(x) < 0 \Leftrightarrow x < -1 \text{ or } 0 < x < 1,$$

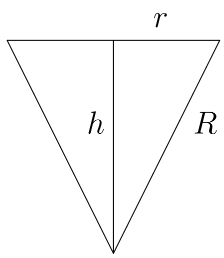
so the graph of f has an inflection point at $(0, 0)$.

8. [avg: 4.5/12] A paper cup in the shape of a cone is made from a circular sheet of paper of radius R by cutting out a sector subtended by an angle θ and then gluing the cut edges of the remaining piece of paper together. What is the maximum volume attainable for the cone? You may assume the volume of a cone with a base of radius r and perpendicular height h is

$$V = \frac{1}{3}\pi r^2 h.$$



Solution: let the resulting conical cup have height h and radius r at the top. A side view of the cup is below. The circumference around the top of the cup is $C = R(2\pi - \theta)$.



In terms of r the circumference of the cup at the top is $C = 2\pi r$. Hence

$$r = R\left(1 - \frac{\theta}{2\pi}\right),$$

for $0 < \theta < 2\pi$. Now $R^2 = h^2 + r^2 \Rightarrow h = \sqrt{R^2 - r^2}$. But it is *not* necessary to put everything in terms of θ; indeed that would be the longest and messiest way to do this problem! The easiest way is to realize that what the problem boils down to is, Maximize the value of $V = \pi r^2 h/3$ given that $r^2 + h^2 = R^2$ is constant. Using $r^2 = R^2 - h^2$, we have

$$V = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2 h - h^3) \Rightarrow \frac{dV}{dh} = \frac{\pi}{3}(R^2 - 3h^2) \Rightarrow \frac{d^2V}{dh^2} = -2\pi h < 0.$$

Hence the critical point is at $3h^2 = R^2$, and the volume of the cup will be maximized if

$$h = \frac{R}{\sqrt{3}}, \quad r = \sqrt{R^2 - h^2} = \sqrt{\frac{2}{3}}\,R \text{ and } V = \frac{2\pi R^3}{9\sqrt{3}}.$$

Alternate Calculation: substitute for h but square V to avoid square roots.

$$V^2 = \frac{\pi^2}{9} r^4 h^2 = \frac{\pi^2}{9} r^4 (R^2 - r^2) = \frac{\pi^2}{9}(r^6 - R^2 r^4) \Rightarrow \frac{dV^2}{dr} = \frac{2\pi^2 r^3}{9}(3r^2 - 2R^2).$$

As before the critical point is at $3r^2 = 2R^2$.

Alternate Solution: put everything in terms of θ but use the chain rule and the facts that

$$\frac{dr}{d\theta} = -\frac{R}{2\pi} < 0 \text{ and/or } \frac{dh}{d\theta} = -\frac{r}{h}\frac{dr}{d\theta} > 0.$$

Then

$$\frac{dV}{d\theta} = \frac{dV}{dh}\frac{dh}{d\theta} = \frac{\pi}{3}(R^2 - 3h^2)\frac{dh}{d\theta} = 0 \text{ if } 3h^2 = R^2.$$

As before, $h = R/\sqrt{3}, r = \sqrt{2}R/\sqrt{3}$. Aside: the ciritical angle is $\theta = 2\pi(\sqrt{3} - \sqrt{2})/\sqrt{3}$, but the angle is not asked for. (See General Comments, if you are interested.)

9. [avg: 7.8/12] Let $f(x) = x^{2/3}$.

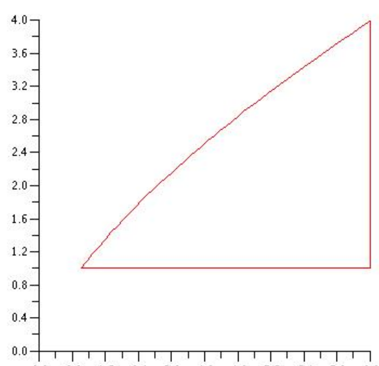
(a) [6 marks] Find the length of the curve $y = f(x)$ for $1 \le x \le 8$.

Solution:

$$\begin{aligned}
L = \int_1^8 \sqrt{1 + (f'(x))^2}\, dx &= \int_1^8 \sqrt{1 + \left(\frac{2}{3x^{1/3}}\right)^2}\, dx \\
&= \int_1^8 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\, dx \\
&= \int_1^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\, dx \\
(\text{let } u = 9x^{2/3}+4;\, du = 6x^{-1/3}\, dx) &= \frac{1}{18}\int_{13}^{40} \sqrt{u}\, du \\
&= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} \\
&= \frac{1}{27}(40\sqrt{40} - 13\sqrt{13})
\end{aligned}$$

(b) [6 marks] Find the volume of the solid of revolution obtained by revolving the region in the xy-plane bounded by the curves $y = f(x), y = 1, x = 1$ and $x = 8$ about the x-axis.

Solution: integrate with respect to x and use the method of washers (discs):



$$\begin{aligned}
V &= \int_1^8 \pi\,((f(x))^2 - 1^2)\, dx \\
&= \int_1^8 \pi\,(x^{4/3} - 1)\, dx \\
&= \pi\left[\frac{3\,x^{7/3}}{7} - x\right]_1^8 \\
&= \frac{332}{7}\pi
\end{aligned}$$

Alternatively: integrate with respect to y and use the method of shells.

$$V = \int_1^4 2\pi y(8-x)\, dy = 2\pi\int_1^4 (8y - y^{5/2})\, dy = 2\pi\left[4y^2 - \frac{2\,y^{7/2}}{7}\right]_1^4 = \frac{332}{7}\pi$$