University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2011**
First Year - CHE, CIV, IND, LME, MEC, MMS
MAT186H1F - CALCULUS I
Exam Type: A

General Comments:

1. Some common misconceptions:

$$\int \ln x \, dx = \frac{1}{x} + C, \ \sec^{-1} y = 1/\cos^{-1} y, \ \int e^y \, dy = \frac{e^y}{y} + C$$

2. A lot of students seem to have totally forgotten what a linear approximation is.

3. Question 5 is an easy *area* problem; many students turned it into an unrelated volume or surface area problem, resulting in some integrals that could not be solved.

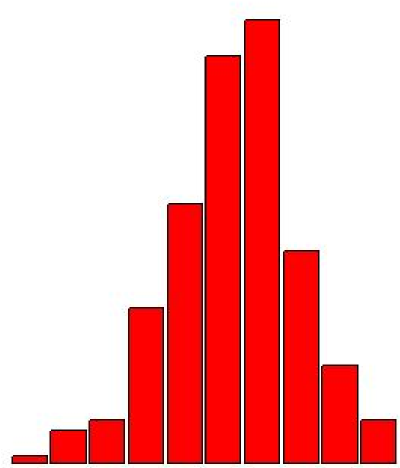
4. Most students could find the intervals on which the graph of Question 6 increases, decreases and is concave up, but very few could calculate the horizontal asymptote or find the removable discontinuities. Moreover, nearly one third of students seemed to think that $\sec^{-1} y = 1/\cos^{-1} y$; the correct connection is $\sec^{-1} y = \cos^{-1}(1/y)$, if $y \neq 0$.

5. Question 7 is a related rates problem, but *first* you have to calculate the volume of the liquid in the bowl, which requires you to use methods of Sections 6.2 or 6.3.

6. Questions 1, 2, 3, 4, 5 and 8 were routine–that's 76% of the exam–and should all have been aced. Nobody should have failed this exam.

Breakdown of Results: 472 students wrote the exam. The marks ranged from 6% to 100%, and the average was 57.2%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.5%
A	8.3%	80-89%	5.7%
B	12.5%	70-79%	12.5%
C	26.1%	60-69%	26.1%
D	23.9%	50-59%	23.9%
F	29.2%	40-49%	15.3%
		30-39%	9.1%
		20-29%	2.5%
		10-19%	1.9%
		0-9%	0.4%



1. [20 marks] Find the following:

(a) [5 marks] $\displaystyle\int \left(\frac{1}{1+x^2} + \tan x + \frac{1}{x}\right) dx$

Solution:

$$\begin{aligned}
\int \left(\frac{1}{1+x^2} + \tan x + \frac{1}{x}\right) dx &= \int \frac{dx}{1+x^2} + \int \frac{\sin x}{\cos x}\, dx + \int \frac{dx}{x} \\
&= \tan^{-1} x - \ln|\cos x| + \ln|x| + C \\
\text{or} &\quad \tan^{-1} x + \ln|\sec x| + \ln|x| + C
\end{aligned}$$

(b) [5 marks] $\displaystyle\lim_{x\to\infty} \left(1 - \frac{3}{x}\right)^x$

Solution: the limit is in the 1^∞ form. Let the limit be L.

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} x \ln\left(1 - \frac{3}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{3}{x}\right)}{\frac{1}{x}} \\
\text{(L'H)} &= \lim_{x\to\infty} \frac{\frac{1}{1-\frac{3}{x}}\left(\frac{3}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{(-3)}{1-\frac{3}{x}} = -3 \\
\Rightarrow L &= e^{-3}
\end{aligned}$$

(c) [5 marks] the linear approximation of $79^{1/4}$, without using your calculator.

Solution: let $f(x) = x^{1/4}; a = 81$. Then $f(a) = 3$ and $f'(a) = \dfrac{1}{4}(81)^{-3/4} = \dfrac{1}{108}$. So

$$79^{1/4} = f(79) \simeq f'(81)(79 - 81) + f(81) = -\frac{1}{54} + 3 = \frac{161}{54}$$

(d) [5 marks] an approximation of the solution to the equation $x^3 + x - 1 = 0$, correct to 2 decimal places. (You will need your calculator.)

Solution: use Newton's method and pick $x_0 = 0.5$. Then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} = \frac{2x_n^3 - x_n + 1}{3x_n^2 + 1};$$

so $x_0 = 0.5 \Rightarrow x_1 = 0.7142857143 \cdots \Rightarrow x_2 = 0.6831797236\ldots$
$\Rightarrow x_3 = 0.6823284233\ldots$ The solution is $x = 0.68$, correct to two decimal places.

2. [10 marks] Find the following:

(a) [5 marks] $\displaystyle\int_0^1 x\sqrt{16+9x^2}\,dx.$

Solution: let $u = 16+9x^2$. Then $du = 18x\,dx$ and

$$\begin{aligned}\int_0^1 x\sqrt{16+9x^2}\,dx &= \frac{1}{18}\int_{16}^{25}\sqrt{u}\,du\\ &= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{16}^{25}\\ &= \frac{1}{27}(125-64)\\ &= \frac{61}{27}\end{aligned}$$

(b) [5 marks] $F'(2)$ if $F(x) = \displaystyle\int_0^{x^2}\sqrt{t^2+9}\,dt.$

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = 2x\sqrt{(x^2)^2+9}.$$

So

$$F'(2) = 4\sqrt{25} = 20.$$

3. [12 marks] Sketch the graph of $y = 4x^{1/3} - x^{4/3}$, labeling all critical points, inflection points, and vertical tangents, if any. You may assume

$$y' = \frac{4}{3\,x^{2/3}} - \frac{4\,x^{1/3}}{3} \text{ and } y'' = -\frac{8}{9\,x^{5/3}} - \frac{4}{9\,x^{2/3}}.$$

Solution: let $y = f(x)$; $f'(x) = \dfrac{4}{3}\dfrac{(1-x)}{x^{2/3}}$. Then

$$f'(x) > 0 \Leftrightarrow 1 - x > 0, x \neq 0 \Leftrightarrow x < 0, 0 < x < 1$$

and

$$f'(x) < 0 \Leftrightarrow 1 - x < 0 \Leftrightarrow x > 1.$$

So $(1, 3)$ is an absolute maximum point. There is a vertical tangent at $x = 0$ since

$$\lim_{x\to 0^+} f'(x) = \infty \text{ and } \lim_{x\to 0^-} f'(x) = \infty.$$

$f''(x) = -\dfrac{4}{9}\dfrac{(2+x)}{x^{5/3}}$. Then

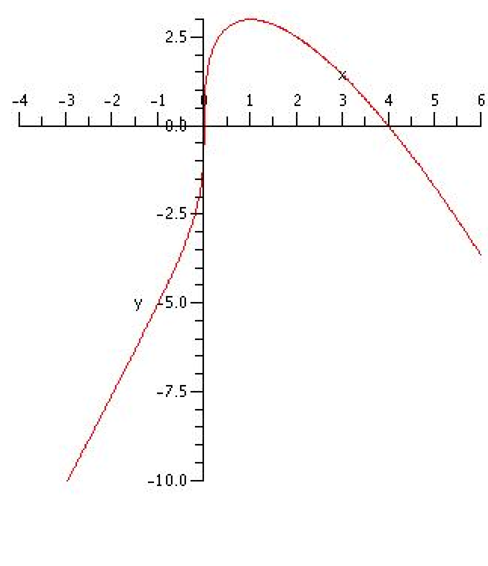
$$f''(x) > 0 \Leftrightarrow \frac{2+x}{x^{5/3}} < 0 \Leftrightarrow -2 < x < 0$$

and

$$f''(x) < 0 \Leftrightarrow \frac{2+x}{x^{5/3}} > 0 \Leftrightarrow x < -2 \text{ or } x > 0.$$

Consequently there are inflection points at $(0, 0)$ and $(-2, -6 \cdot 2^{1/3})$.

The graph of f is shown below:



4. [10 marks] The velocity of a particle at time t is given by $v = 3t^2 - 3$. Find

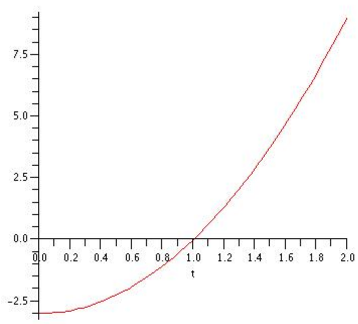
(a) [4 marks] the average velocity of the particle for $0 \leq t \leq 2$.

Solution: average velocity is

$$\frac{1}{2-0}\int_0^2 v\,dt = \frac{1}{2}\int_0^2 (3t^2 - 2)\,dt = \frac{1}{2}[t^3 - 3t]_0^2 = \frac{1}{2}(8-6) = 1.$$

(b) [6 marks] the average speed of the particle for $0 \leq t \leq 2$.

Solution: this is a little trickier since v changes sign at $t = 1$. See graph below.



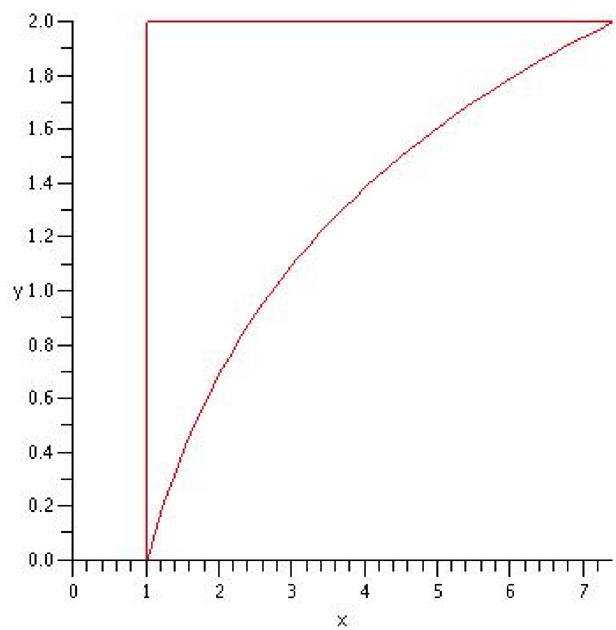
The average speed is

$$\begin{aligned}\frac{1}{2-0}\int_0^2 |v|\,dt &= \frac{1}{2}\int_0^1 (-v)\,dt + \frac{1}{2}\int_1^2 v\,dt\\ &= \frac{1}{2}[3t - t^3]_0^1 + \frac{1}{2}[t^3 - 3t]_1^2\\ &= \frac{2}{2} + \frac{1}{2}(8 - 6 - 1 + 3)\\ &= 1 + 2 = 3\end{aligned}$$

5. [12 marks] Let A be the area of the region in the xy-plane bounded by $x = 1, x = e^2, y = 2$ and $y = \ln x$.

(a) [8 marks] Write down two integrals, one with respect to x and one with respect to y, that both give the value of A.

Solution: the region A is indicated in the graph below.



With respect to x:

$$A = \int_1^{e^2} (2 - \ln x)\, dx.$$

With respect to y:

$$A = \int_0^2 (e^y - 1)\, dy,$$

since $y = \ln x \Leftrightarrow x = e^y$.

(b) [4 marks] Find the value of A.

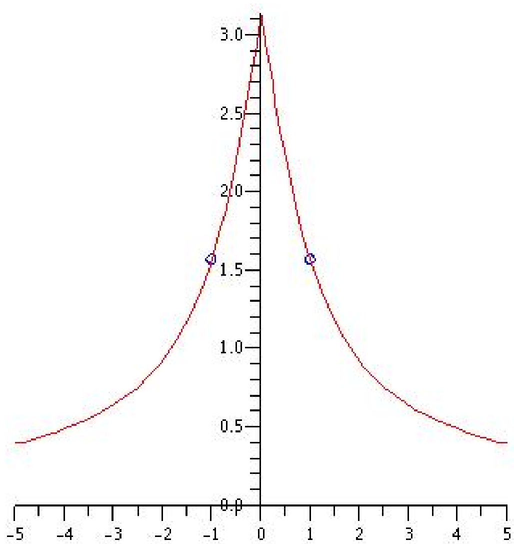
Solution: integrate with respect to y, since at this stage of the game, we don't know how to integrate $\ln x$ with respect to x.

$$A = \int_0^2 (e^y - 1)\, dy = [e^y - y]_0^2 = e^2 - 2 - 1 + 0 = e^2 - 3.$$

6. [12 marks] Sketch the graph of $y = \sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$, labeling all critical points, inflection points, vertical tangents, asymptotes and discontinuities, if any. You may assume

$$y' = -\frac{2x}{(x^2+1)\,|x|} \text{ and } y'' = \frac{4\,|x|}{(x^2+1)^2}.$$

Solution: this question looks worse than it is. For one thing, the graph of y is symmetric with respect to the y-axis, so you only have to analyze the curve for $x \geq 0$. Note that the graph is always concave up, is decreasing for $x > 0, x \neq 1$, and that the only critical point is $(0, \pi)$. There are no inflection points. The rest of the details:



$x = 0$ is not a vertical tangent, since

$$\lim_{x\to 0^+} f'(x) = -2.$$

$y = 0$ is a horizontal asymptote since

$$\begin{aligned} &\lim_{x\to\infty} \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right) \\ =\ &\sec^{-1}\left(\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)\right) \\ =\ &\sec^{-1} 1 = 0 \end{aligned}$$

There are removable discontinuities at $(\pm 1, \pi/2)$ since $\dfrac{x^2+1}{x^2-1}$ is not defined at $x = \pm 1$ but both

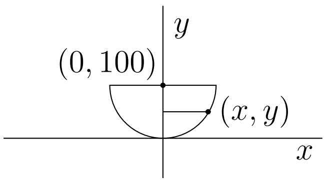
$$\lim_{x\to 1^+} \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right) = \lim_{u\to\infty} \sec^{-1} u = \frac{\pi}{2}$$

and

$$\lim_{x\to 1^-} \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right) = \lim_{u\to-\infty} \sec^{-1} u = \frac{\pi}{2}.$$

7. [12 marks] If water enters a hemispherical bowl of radius 100 cm at a rate of 10 cm^3/sec, how fast will the water level be rising when the depth of the water in the bowl is 50 cm?

Solution: need to calculate the volume of the bowl from its bottom to a depth of h.



The diagram to the left shows a side view of the bowl; the equation of the bowl's profile is

$$x^2 + (y - 100)^2 = 100^2$$

Thus $x^2 = 200y - y^2$.

Use the method of discs, integrating with respect to y, to obtain the volume of the bowl from its bottom to a depth of h.

$$V = \int_0^h \pi x^2 \, dy = \int_0^h \pi(200y - y^2) \, dy.$$

By the chain rule, and the Fundamental Theorem of Calculus,

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = \pi(200h - h^2)\,\frac{dh}{dt}.$$

Substitute

$$\frac{dV}{dt} = 10, \;\; h = 50$$

and solve for $\dfrac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{1}{\pi}\frac{10}{200 \cdot 50 - 50^2} = \frac{1}{750\pi}.$$

Thus when the depth of the water in the bowl is 50 cm, the water level is rising at a rate of $1/(750\pi)$ cm per sec.

8. [12 marks] Let $f(x) = x^{3/2}$.

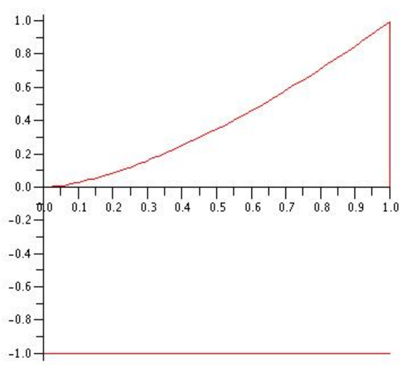
(a) [6 marks] Find the length of the curve $y = f(x)$ for $0 \le x \le 1$.

Solution:

$$\begin{aligned} L = \int_0^1 \sqrt{1+(f'(x))^2}\,dx &= \int_0^1 \sqrt{1+\left(\frac{3\sqrt{x}}{2}\right)^2}\,dx \\ &= \frac{1}{2}\int_0^1 \sqrt{4+9x}\,dx \\ \text{(let } u = 4+9x\text{)} &= \frac{1}{8}\int_4^{13} \sqrt{u}\,du \\ &= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{13} \\ &= \frac{1}{27}(13^{3/2}-8) \end{aligned}$$

(b) [6 marks] Find the volume of the solid of revolution obtained by revolving the region in the xy-plane bounded by the curves $y = f(x), y = 0, x = 0$ and $x = 1$ about the line $y = -1$.

Solution: integrate with respect to x and use the method of washers (discs):



$$\begin{aligned} V &= \int_0^1 \pi\left((f(x)+1)^2 - 1^2\right)dx \\ &= \int_0^1 \pi\left(x^3 + 2x^{3/2}\right)dx \\ &= \pi\left[\frac{x^4}{4} + \frac{4x^{5/2}}{5}\right]_0^1 \\ &= \frac{21}{20}\pi \end{aligned}$$

Alternatively: integrate with respect to y and use the method of shells.

$$V = \int_0^1 2\pi(y+1)(1-y^{2/3})\,dy = 2\pi\int_0^1 (y - y^{5/3} + 1 - y^{2/3})\,dy = \frac{21}{20}\pi$$